Annals of Mathematical Logic 12 (1977) 25–58
© North-Holland Publishing Company

FLIPPING PROPERTIES: A UNIFYING THREAD IN THE THEORY OF LARGE CARDINALS

F.G. ABRAMSON, L.A. HARRINGTON, E.M. KLEINBERG, and W.S. ZWICKER
Massachusetts Institute of Technology, Cambridge, MA 02139, U.S.A.

Received 16 June 1975

The theory of large cardinals has in recent years centered itself principally around two notions. The first of these, that of *completely additive measure*, initially gained prominence with Scott's celebrated theorem that the existence of a measurable cardinal implies the existence of nonconstructible sets. And the second key notion, that of *partition property*, became well known soon thereafter via a striking improvement of Scott's result, namely Rowbottom's theorem that the existence of a Ramsey cardinal implies the existence of a nonconstructible set of integers.[1] Now although the statements of these two primary results are strongly related, the methods used in proving them are entirely different. In fact, these two methods served as the foundations for divergent paths from which modern large cardinal theory subsequently developed.

One of the principal reasons for the disparity between the "measure" approach and the "partition" approach to large cardinals is the fact that although there are many simple and effective combinatorial ways to weaken and modify partition relations, the same is not true with respect to measure properties. The known methods for weakening the notion of measurable cardinal were either strictly model theoretic or were extremely narrow and restricted.

In this paper we attack the problem of modifying full measurability. We shall introduce a class of purely combinatorial properties of cardinals known as flipping properties and show them to unify a great many previously divergent ideas from large cardinal theory.

It would be unrealistic during this general discussion to go into any great detail concerning the specifics of flipping properties but from a crude motivational point of view, perhaps this will help: suppose that K is an uncountable cardinal. Then if A is a given subset of K, the "flip" of A means simply the complement of A, $K - A$. But more to the point, if $A_0, A_1, A_2, A_3, A_4, A_5, \ldots, A_\alpha, \ldots$ is a given sequence of subsets of K, then a "flip" of that sequence is a sequence of the form $B_0, B_1, B_2, B_3, B_4, B_5, \ldots, B_\alpha, \ldots$ where for each α, B_α is either A_α or the flip of A_α.

[1] Both these notions were already well-known to combinatoricists. These results, however, captured the attention of logicians interested in the model theoretic properties of set theory.

Flipping properties, now, can be described roughly as properties of K involving assertions of the form given a sequence of subsets of K of length γ there exists a flip of the sequence such that the intersection of the sets in the flip has cardinality λ. Here are three specific examples:

Property 1. For any $\alpha < K$ and any sequence s of subsets of K of length α there exists a flip of s such that the intersection of the sets in the flip has cardinality K.

Property 2. Given any sequence s of subsets of K of length K, there exists a flip of the sequence such that the intersection of any fewer than K-many sets in the flip has cardinality K.

Property 3. Given any sequence s of subsets of K, there exists a flip of s such that the intersections of any fewer than K-many sets in the flip has cardinality K.

Now it turns out that these three flipping properties are equivalents to well-known large cardinal properties. The first characterizes K's being strongly inaccessible, the second K's satisfying $K \to (K)^2$, and the third K's being measurable.

As this paper proceeds we shall, in addition to proving these three equivalences and others like them, present a sampling of the theory and techniques of flipping properties. This will involve a presentation both of a number of new theorems (combinatorial ones as well as noncombinatorial ones) as well as a number of simplified flipping-style proofs of older theorems.

Our sampling will by no means be complete but rather will be designed to convey some of the power behind the ideas. For further results one should consult the list of references at the end of the paper.

1. Combinatorial consequences of flipping properties

We begin with a simple "flipping" characterization of strong inaccessibility. Then, as successively more uniformity is added to the flipping assumptions, we will be led quite naturally to properties equivalent to various well known large cardinal hypotheses. Thus the "flipping" idea will show how this array of previously known hypotheses actually forms a coherent progression.

A few notations useful in discussing flipping: Throughout this paper, K will always denote an uncountable cardinal. S_K will denote $\{f \mid f: K \to 2^K\}$, i.e., the set of all K-sequences of subsets of K. s_K will denote $\{f \mid f: \alpha \to 2^K, \alpha < K\}$, i.e., the set of all less-than-K-sequences of subsets of K. For $t, t' \in S_K$ we write $t \sim t'$ to mean:

$$(\forall \beta < K)(t'(\beta) \in \{t(\beta), K - t(\beta)\}),$$

i.e., t' is a "flip" of t: in every place of the sequence t, t' chooses either the subset of K given by t or "flips" that subset, taking its complement. Clearly $\sim$ is an equivalence relation. For $t, t' \in s_\kappa$, $\sim$ is defined similarly, i.e., $t \sim t'$ means domain t = domain t' and

$$(\forall \beta \in \text{domain } t)(t'(\beta) \in \{t(\beta), \kappa - t(\beta)\}).$$

for $t \in S_\kappa$ or s_κ, $\bigcap t$ denotes $\bigcap\{t(\beta) \mid \beta \in \text{domain } t\}$.

1.1. *Flips which give large intersections*

The following theorems are presented as a gentle introduction to the use of flipping properties.

Theorem 1.1.1. *κ is strongly inaccessible iff*

$$(\forall t \in s_\kappa)(\exists t')(t' \sim t \text{ and } \overline{\overline{(\bigcap t')}} = \kappa)$$

(The latter property says that any less-than-κ sequence can be flipped so as to have size-κ intersection.)

Proof. ($\Rightarrow$) Say domain $t = \alpha$. Let F = set of flips of $t = \{u \mid u \sim t\}$. $\overline{\overline{F}} = 2^\alpha < \kappa$, and $\bigcup_{u \in F} \bigcap u = \kappa$. (Indeed, for each $\beta \in \kappa$ there is a unique $u \in F$ such that $\beta \in \bigcap u$.) Thus, by regularity, for some $u \in F$, $\overline{\overline{(\bigcap u)}} = \kappa$. Let t' be any such u.

($\Leftarrow$) If κ *is not regular, having an unbounded* α-sequence, let $\{\beta_\xi \mid \xi < \alpha < \kappa\}$ be cofinal in κ. Let $t(\xi) = \kappa - \beta_\xi$ for $\xi < \alpha$. $t(\xi)$ is the final segment of κ above β_ξ. Now suppose $t' \sim t$. If $t' = t$, then $\bigcap t' = 0$; but if $t'(\xi) \neq t(\xi)$ for some $\xi < \alpha$, then $t'(\xi) = \beta_\xi$, so $\bigcap t' \subseteq \beta_\xi$ and therefore $\overline{\overline{(\bigcap t')}} < \kappa$.

If $2^\alpha \geq \kappa$, $\alpha < \kappa$, let $\underset{\sim}{\beta} : \kappa \to^{1\text{-}1} 2^\alpha$, i.e., for $\beta \in \kappa$, $\underset{\sim}{\beta} : \alpha \to 2$. For $\xi < \alpha$ let $t(\xi) = \{\beta \mid \underset{\sim}{\beta}(\xi) = 1\}$. Take any t' such that $t' \sim t$. $\bigcap t'$ can have at most one element, since for any $\beta \in \bigcap t'$ we have $(\forall \xi < \alpha)(\underset{\sim}{\beta}(\xi) = 1 \leftrightarrow t(\xi) = t'(\xi))$, so any $\beta, \gamma \in \bigcap t'$ have $\underset{\sim}{\beta} = \underset{\sim}{\gamma}$, and $\underset{\sim}{\beta}$ is one-one.[2] □

Our proof actually shows that any flip of fewer than κ-many subsets of κ which has large intersection can be extended to include a few more sets. We have

Corollary 1.1.2. *Let κ be strongly inaccessible. Then if $v \in s_\kappa$ and $\overline{\overline{(\bigcap v)}} = \kappa$ and if $t \in s_\kappa$ then there exists $t' \sim t$ such that $\overline{\overline{(\bigcap vt')}} = \kappa$ (vt' denotes the concatenation of the two sequences).*

Proof. Let $F = \{u \mid u \sim t\}$. Then as in Theorem 1.1.1, since $(\bigcup_{u \in F} \bigcap u) \cap (\bigcap v) = \kappa$, there is a $u \in F$ such that $\overline{\overline{((\bigcap u) \cap (\bigcap v))}} = \kappa$. Take t' = such a u. □

This observation will be important in the section of this paper which deals with the construction of outside measures.

[2] The ideas used in this proof easily give an interesting characterization of the Continuum Hypothesis:

$$2^{\aleph_0} > \aleph_1 \leftrightarrow (\forall t : \aleph_1 \to 2^{\aleph_0})(\exists t')(t' \sim t \text{ and } \bigcap t' = 0).$$

We have seen that a flipping property of short sequences is equivalent to strong inaccessibility. A more uniform flipping property yields the weakly compact cardinals.

Theorem 1.1.3. *K is weakly compact iff*

$$(\forall t \in S_K)(\exists t')(t' \sim t \text{ and } \forall \delta < K, (\bigcap t''\delta)^{=} = K).$$

(This property says that a K-sequence can be flipped so that all initial sequences have size-K intersection.)

Proof. ($\Longrightarrow$) We take as our definition of weakly compact the partition relation $K \to (K)^2$.[3] We will use the well known results that

$$K \to (K)^2 \leftrightarrow \forall m, n \in \omega (K \to (K)^m_n)$$

$$\leftrightarrow (K \text{ is strongly inaccessible and has the tree property}).$$

Let $t \in S_K$. Let $F: [K]^3 \to 2$ by

$$F(\{\alpha, \beta, \gamma\}_<) = \begin{cases} 0 & \text{if } \{\delta < \alpha \mid \beta \in t(\delta)\} = \{\delta < \alpha \mid \gamma \in t(\delta)\}, \\ 1 & \text{otherwise.} \end{cases}$$

Since K is weakly compact, we can find $H \subseteq K$, $\bar{\bar{H}} = K$, such that $(\forall x \in [H]^3)(F(x) = 0)$ or $(\forall x \in [H]^3)(F(x) = 1)$. Suppose the second holds. Let α be the least element of H. Then for $\beta, \gamma \in H$, $\alpha < \beta < \gamma$, we have

$$\{\delta < \alpha \mid \beta \in t(\delta)\} \neq \{\delta < \alpha \mid \gamma \in t(\delta)\},$$

thus giving rise to K-many distinct subsets of α, a contradiction as K is strongly inaccessible. So $\forall x \in [H]^3, F(x) = 0$. We can define, then,

$$t'(\xi) = \begin{cases} t(\xi) & \text{if } \beta \in t(\xi), (\forall \beta \in H, \beta > (\text{least } \alpha \in H \text{ such that } \alpha > \xi)), \\ K - t(\xi) & \text{iff } \beta \notin t(\xi), (\forall \beta \in H, \beta > (\text{least } \alpha \in H \text{ such that } \alpha > \xi)). \end{cases}$$

Clearly $t' \sim t$ and for any $\delta < K$, $\bigcap t''\delta \supseteq H - (\text{least } \alpha \in H \text{ such that } \alpha > \delta)$.

($\Longleftarrow$) We prove K is strongly inaccessible and has the tree property. The former is immediate since this flipping property is stronger than the previous one which is equivalent to strong inaccessibility. Now, suppose $\langle T, \leqslant \rangle$ is a tree of height K such that for each $\alpha < K$, T_α, the α^{th} level of T, has $(T_\alpha)^{=} < K$. We need a path in T of length K. Since $\bar{\bar{T}} = K$ we may assume for ease of notation that $T = K$ For each $\alpha < K$, let $t(\alpha) = \{\beta \mid \alpha \leqslant \beta\}$. Take t' such that $t' \sim t$ and $\forall \delta < K$ $(\bigcap t''\delta)^{=} = K$. Let $P = \{\alpha \mid t'(\alpha) = t(\alpha)\}$. ($P$ will be the path through the tree.) Now, for each $\delta < K$ there is a unique $\alpha \in T_\delta \cap P$. Why? If no $\alpha \in T_\delta \cap P$ then

[3] See (Kleinberg [16]) for definitions and background on such partition relations. We say "weakly compact" for brevity; these cardinals are sometimes called "strongly inaccessible weakly compact".

$\bigcap_{\alpha\in T_\delta} t'(\alpha) = \bigcap_{\alpha\in T_\delta}\{\beta \mid \alpha \leqslant \beta\} = \bigcup_{\xi<\delta} T_\xi$. Therefore $(\bigcap_{\alpha\in T_\delta} t'(\alpha))^{=} < K$, contradicting the property of t'. If $\alpha \neq \alpha'$, $\alpha, \alpha' \in T_\delta \cap P$, then

$$t'(\alpha) \cap t'(\alpha') = \{\beta \mid \alpha \leqslant \beta\} \cap \{\beta \mid \alpha' \leqslant \beta\} = \emptyset$$

since T is a tree. Once again, this contradicts that $(t'(\alpha) \cap t'(\alpha'))^{=} = K$. It is now clear that P is the desired path, for if $\delta < \delta' < K$, letting $\{\alpha\} = T_\delta \cap P$, $\{\alpha'\} = T_{\delta'} \cap P$, were $\alpha \not\leqslant \alpha'$ we would have $t'(\alpha) \cap t'(\alpha') = \emptyset$. □

As in the case of strongly inaccessible cardinals there is a corollary to our proof which will be important in a later section of this paper. To state this corollary it will be convenient to think in terms of collections of subsets of K rather than sequences of subsets, and to speak of flips of these collections.

Let $C \subseteq 2^K$. We say C' is a flip of C, also written $C' \sim C$, if $(\forall x \in C)(x \in C'$ or $K - x \in C')$ and $(\forall x \in C')(x \in C$ or $K - x \in C)$. Again $\sim$ is an equivalence relation. For K weakly compact, Theorem 1.1.3 shows that

$$(\forall C \subseteq 2^K, \bar{\bar{C}} \leqslant K)(\exists C')[C' \sim C \text{ and } (\forall \mathscr{C} \subseteq C', \bar{\bar{\mathscr{C}}} < K)((\bigcap \mathscr{C})^{=} = K)].$$

The proof actually shows that having taken any C for a given C we can then be given a $\mathscr{D} \subseteq 2^K$, $\mathscr{D} \leqslant K$, and find a $\mathscr{D}' \sim \mathscr{D}$ such that

$$(\forall \mathscr{C} \subseteq C' \cup \mathscr{D}', \bar{\bar{\mathscr{C}}} < K), \qquad (\bigcap \mathscr{C})^{=} = K.$$

Why? Suppose $C' \sim C$ and $(\forall \mathscr{C} \subseteq C', \bar{\bar{\mathscr{C}}} < K), (\bigcap \mathscr{C})^{=} = K$. Let t' be such that $t''K = C'$. Define $H \subseteq K$, $H = \{h_\alpha \mid \alpha < K\}$ by $h_\alpha =$ least γ such that $\gamma \geqslant \alpha$ and $\gamma \in \bigcap t''\alpha$. Then $\bar{\bar{H}} = K$ and for any $\delta < K$, $\{h_\alpha \mid \alpha > \delta\} \subseteq t'(\delta)$. So H has a property like the diagonal intersection[4] of t', i.e., for any $\delta < K$, $(\exists \rho < K)$ $(\bigcap t''\delta \supseteq H - \rho)$. (For a true diagonal intersection we would have $\rho = \delta$.)

Now we can repeat the proof of 1.1.3 relativizing everything to this pseudo-diagonal intersection:

Given $\mathscr{D}$, take $u \in S_K$ such that $u''K = \mathscr{D}$, define $F: [H]^3 \to 2$ by

$$F(\{\alpha, \beta, \gamma\}_<) = \begin{cases} 0 & \text{if } \{\delta < \alpha \mid \delta \in u(\beta)\} = \{\delta < \alpha \mid \delta \in u(\gamma)\}, \\ 1 & \text{otherwise,} \end{cases}$$

and let $H' \subseteq H$, $\bar{\bar{H}}' = K$, H' homogeneous for F. As before the homogeneity goes to 0, so defining

$$u'(\xi) = \begin{cases} t(\xi) & \text{if } \beta \in t(\xi), (\forall \beta \in H', \beta > (\text{least } \alpha \in H' \text{ such that } \alpha > \xi)), \\ K - t(u) & \text{if } \beta \notin t(\xi), (\forall \beta \in H', \beta > (\text{least } \alpha \in H' \text{ such that } \alpha > \xi)), \end{cases}$$

[4] The diagonal intersection of t', $\Delta t'$, is defined as

$$\{\beta \mid 0 < \beta < K \text{ and } (\forall \alpha < \beta)(\beta \in t'(\alpha))\},$$

$(= \{\beta \mid 0 < \beta < K, \beta \in \bigcap t''\beta\})$. We have chosen to rule out 0 from membership in $\Delta t'$ by fiat, mainly to allow a clean statement of 1.4.2.

we have for each $\delta < K$,

$$\bigcap_{\xi<\delta} u'(\xi) \supseteq H' - \rho \qquad \text{for some } \rho < K.$$

Now let $\mathscr{D}' = u'''K$. If $\mathscr{E} \subseteq C' \cup \mathscr{D}'$ and $\bar{\bar{\mathscr{E}}} < K$, then $\bigcap \mathscr{E} = (\bigcap (C' \cap \mathscr{E})) \cap (\bigcap (\mathscr{D}' \cap \mathscr{E}))$, which contains $H' = \theta$ for some $\theta < K$ since

$\bigcap (C' \cap \mathscr{E})$ contains a final segment of H,

$\bigcap (\mathscr{D}' \cap \mathscr{E})$ contains a final segment of H'.

In summary we have

Corollary 1.1.4. *If $C \subseteq 2^K$, $\bar{\bar{C}} \leq K$, then*

$$\exists C' \sim C((\forall \mathscr{E} \subseteq C', \bar{\bar{\mathscr{E}}} < K)(\bigcap \mathscr{E})^{=} = K).$$

Further, given such a C' we have

$$(\forall \mathscr{D} \subseteq 2^K, \bar{\bar{\mathscr{D}}} \leq K)(\exists \mathscr{D}' \sim \mathscr{D})(\forall \mathscr{E} \subseteq C' \cup \mathscr{D}', \bar{\bar{\mathscr{E}}} < K ((\bigcap \mathscr{E})^{=} = K).[5]$$

1.2. Flips which give large diagonal intersection

We can see that there was more uniformity inherent in the flipping definition of weak compactness than appeared on the surface. Not only could we find a flip such that $\bigcap t''' \delta$ was large, there was, in fact, a single large set H such that $\forall \delta (\bigcap t''' \delta \supseteq H - \rho(\delta))$ where $\rho: K \to K$. An even greater uniformity would be that ρ is the identity. Looked at differently, a weakly compact cardinal has a flipping property that gives a large set similar to a diagonal intersection. A stronger property would say that the diagonal intersection itself is large. We are led to investigate the following property:

$$(\forall t \in S_K)(\exists t')(t' \sim t \text{ and } (\Delta t')^{=} = K).$$

This property is equivalent to the well-known property of *weak ineffability* whose model theoretic consequences we will explore later.[6]

Theorem 1.2.1. *K is weakly ineffable iff*

$$(\forall t \in S_K)(\exists t')(t' \sim t \text{ and } (\Delta t')^{=} = K).$$

(This property says any K-sequence can be flipped to yield K-size diagonal intersection.)

Proof. ($\Rightarrow$) Given t, define $A_\alpha = \{\beta < \alpha \mid \alpha \in t(\beta)\}$. Let $H \subseteq K$ have $\bar{\bar{H}} = K$ and take $A \subseteq K$ such that $(\forall \alpha \in H)(A \cap \alpha = A_\alpha)$. Let

[5] Kunen [21] worked with the following characterization of K being weakly compact. Given $\mathscr{B} \subseteq 2^K$, $\bar{\bar{\mathscr{B}}} = K$, $\mathscr{B}$ a K-complete Boolean algebra, then any K-additive filter on $\mathscr{B}$ can be extended to a K-additive ultrafilter on $\mathscr{B}$. This property is an immediate consequence of Corollary 1.1.4; take C' = the filter and $\mathscr{D} = \mathscr{B} - C'$.

[6] Recall, a cardinal K is weakly ineffable iff given for each $\alpha < K$ an $A_\alpha \subseteq \alpha$ we can find $H \subseteq K$, $\bar{\bar{H}} = K$, and $A \subseteq K$ such that $(\forall \alpha \in H)(A \cap \alpha = A_\alpha)$. (This says that a K-sequence of subsets can be made to cohere in K-many places.)

$$t'(\xi)=\begin{cases} t(\xi) & \text{if } \xi \in A, \\ K - t(\xi) & \text{if } \xi \notin A. \end{cases}$$

$t' \sim t$ and $H \subseteq \Delta t'$ since for any $\alpha \in H$, if $\xi < \alpha$ either

(i) $t'(\xi) = t(\xi) \Longrightarrow \xi \in A \Longrightarrow \xi \in A_\alpha \Longrightarrow \alpha \in t(\xi) \Longrightarrow \alpha \in t'(\xi)$,

or

(ii) $t'(\xi) = K - t(\xi) \Longrightarrow \xi \notin A \Longrightarrow \xi \notin A_\alpha \Longrightarrow \alpha \notin t(\xi) \Longrightarrow \alpha \in t'(\xi)$,

so we have $\alpha \in t'(\xi)$ for all ξ less than α.

($\Longleftarrow$) Given $A_\alpha \subseteq \alpha$ let $t(\xi) = \{\alpha \mid \xi \in A_\alpha\}$. Let $t' \sim t$ be such that $(\Delta t')^{=} = K$. Let $H = \Delta t'$. Then if $\alpha < \beta$, $\alpha, \beta \in H$, we have for any $\xi < \alpha$ that $t'(\xi) = t(\xi) \Longleftrightarrow \alpha \in t(\xi) \Longleftrightarrow \xi \in A_\alpha$ and for any $\xi < \beta$ that $t'(\xi) = t(\xi) \Longleftrightarrow \beta \in t(\xi) \Longleftrightarrow \xi \in A_\beta$.

Therefore, $A_\beta \cap \alpha = \{\xi < \alpha \mid t'(\xi) = t(\xi)\} = A_\alpha$.

Now take $A = \bigcup_{\alpha \in H} A_\alpha$. For $\alpha \in H$, clearly $A_\alpha = A \cap \alpha$. □

1.3. *Flips which give stationary diagonal intersection*

What is a flipping-type characterization of ineffable[7] cardinals? Well, the proof above will show:

Corollary 1.3.1. *K is ineffable iff*

$$(\forall t \in S_K)(\exists t')[t' \sim t \text{ and } \Delta t' \text{ is stationary}]$$

(The flipping property says that every sequence can be flipped so as to have a stationary diagonal intersection.)

However, like the many other well known formulations of ineffability, the above characterization uses the notion of stationary set. Much more interesting is the next theorem which dispenses with all mention of this notion, and which shows how we reach this cardinal by once again adding more uniformity to a flipping property. Here is the idea: the diagonal intersection of a sequence of sets depends on the order of the sets, not just on what sets appear in the sequence. With a weakly ineffable cardinal we can flip a given sequence so that its diagonal intersection is large. Were the sequence rearranged, a different flip might be required. A more uniform property would be that we could flip the sets such that *any* arrangement of them in a K-sequence would have large intersection.

Theorem 1.3.2. *K is ineffable iff*

$$(\forall t \in S_K)\exists t'[t' \sim t \text{ and } (\forall \pi : K \to K)(\Delta(t' \circ \pi))^{=} = K)].$$

[7] Recall, K is ineffable if, given for each $\alpha < K$ an $A_\alpha \subseteq \alpha$, we can find a stationary set H and an $A \subseteq K$ such that $(\forall \alpha \in H)[A \cap \alpha = A_\alpha]$, i.e., subsets of K can be made to cohere on a stationary set, not just on a set of full cardinality. ($H \subseteq K$ is called "stationary" if it has non-empty intersection with every closed unbounded subset of K. $C \subseteq K$ is "closed" if $\forall$ limit $\alpha < \kappa$, if $\alpha = \bigcup (C \cap \alpha)$ then $\alpha \in C$.) This stronger coherence property has greater model theoretic consequences than weak ineffability.

Proof. ($\Longrightarrow$) Let $t' \sim t$ be such that $\Delta t'$ is stationary. Now we can use the following key lemma.

Lemma 1.3.3. *Let η be an uncountable regular cardinal. Let $s, s' \in S\eta$ and suppose $s'''\eta \subseteq s''\eta$. Then there is a closed unbounded subset, C, of η with $\Delta s' \supseteq C \cap \Delta s$.*

(In other words, if complements of closed unbounded sets are thought of as null sets, one cannot shrink a diagonal intersection by rearranging the order in which it is taken.)

Proof of Lemma. Define $f: \eta \to \eta$ by

$$f(\alpha) = \text{least } \beta \geqslant \alpha \text{ such that } (\forall \gamma < \alpha)(\exists \delta < \beta)[s'(\gamma) = s(\delta)].$$

f is clearly a continuous, non-decreasing, unbounded function, so we can take C to be a closed, unbounded subset of η such that $(\forall \alpha \in C)[f(\alpha) = \alpha]$. For $\alpha \in C \cap \Delta s$ we have $(\forall \gamma < \alpha)(\exists \delta < \alpha)[s'(\gamma) = s(\delta)]$ and $(\forall \delta < \alpha)[\alpha \in s(\delta)]$, so $(\forall \gamma < \alpha)[\alpha \in s'(\gamma)]$, i.e., $\alpha \in \Delta s'$. $\square$

Now let $\pi: K \to K$. As $(t' \circ \pi)'' K \subseteq t''' K$, let C be a closed unbounded subset of K such that $\Delta(t' \circ \pi) \supseteq C \cap \Delta t'$. Since C is closed unbounded and $\Delta t'$ is stationary, $C \cap \Delta t'$ is stationary, so $\Delta(t' \circ \pi)$ is stationary and must therefore have cardinality K.

($\Longleftarrow$) Given $t \in S$, we need $t' \sim t$ with $\Delta t'$ stationary. Take $u \in S_K$ with range $t'' K \cup \{K - \alpha \mid \alpha < K\}$. (We have just enlarged the collection of subsets by throwing in all "final segments" of K.) Let $u' \sim u$ be such that

$$(\forall \pi: K \to K)[(\Delta(u' \circ \pi))^{=} = K].$$

Take $t' \sim t$ as determined by requiring that $t''' K \subseteq u''' K$. Let C be any closed unbounded subset of K and suppose $\Delta t' \cap C = \emptyset$. Set $C' =$ all limit ordinals in C. Then C' is also closed unbounded and $\Delta t' \cap C' = \emptyset$. Define $v \in S_K$ by

$$v(\alpha + 1) = t'(\alpha) \quad \text{for } \alpha < K,$$

$$v(\lambda) = K - (\text{least element of } C' > \lambda) \quad \text{for } \lambda \text{ a limit.}$$

Since $K - \alpha \in u''' K$ for each $\alpha < K$, $v = u' \circ \pi$ for some $\pi: K \to K$, so $(\Delta v)^{=} = K$. But we can see that $\Delta v = \emptyset$ as follows: if $\xi \in C'$, ξ is a limit ordinal, so $v'' \xi \supseteq t''' \xi$. As $\xi \notin \bigcap t''' \xi$, $\xi \notin \bigcap v'' \xi$, i.e., $\xi \notin \Delta v$. If $\xi \notin C'$, as C' is closed we can take $\lambda =$ greatest $\alpha \in C'$ with $\alpha < \xi$. λ is a limit ordinal $< \xi$ and $\xi \notin v(\lambda)$, so $\xi \notin \Delta v$. We've seen $\Delta v = \emptyset$ and this contradiction completes the theorem. $\square$

We have exhibited two flipping properties equivalent to ineffability. There is, in addition, an equivalent partition property. $\alpha \to_s (\alpha)^\beta_\gamma$ denotes the partition property $\alpha \to (\alpha)^\beta_\gamma$ with the additional requirement that the homogeneous set can be chosen to be stationary.

Theorem 1.3.4. (Jensen–Kunen [13]). *K is ineffable iff $K \rightarrow_s (K)^2$.*

Proof. ($\Longrightarrow$) Let $F: [K]^2 \rightarrow 2$. We build a tree T *à la* Ramsey's theorem for F, then use the flipping property

$$(\forall t \in S_K)(\exists t')[t' \sim t \text{ and } \Delta t \text{ is stationary}]$$

to get a stationary path through the tree. T is defined inductively by levels, T_ξ, $\xi < K$. The nodes of T are subsets of K. One node, A, is a successor of another, B, iff $A \subseteq B$. The bottom level, T_0, has one node, K itself. Given T_α, construct $T_{\alpha+1}$ as follows: For each $A \in T_\alpha$ let x_A be the least element of A. We put two nodes, A_0 and A_1 into $T_{\alpha+1}$ where

$$A_i = \{y \in A \mid y \neq x_A \text{ and } F(\{x_A, y\}) = i\}.$$

For λ a limit we put a non-empty set A on level λ if there is a path P through $\bigcup_{\beta<\lambda} T_\beta$, $P = \{A_\delta \mid \delta < \lambda\}$, such that $A = \bigcap_{\delta<\lambda} A_\delta$. It is easy to see, using the inaccessibility of K, that for each $\alpha < K$, $\bigcup T_\alpha \supseteq K - \rho$ for some $\rho < K$, that T has height K and each level has fewer than K nodes. Letting A_α be the unique node, A, of T such that α is the least element of A, then $\{A_\alpha \mid \alpha < K\}$ enumerates T, and for all α, $A_\alpha \in T_\beta$ for some $\beta \leq \alpha$. Define another tree, $V = \langle K, <^* \rangle$, where $\alpha <^* \beta$ if $\beta \in A_\alpha$. Then $\langle K, <^* \rangle \cong \langle T, \supseteq \rangle$.

Now, as in the proof of Theorem 1.1.3, let $t(\alpha) = \{\beta \mid \alpha <^* \beta\}$ and take $t' \sim t$ such that $\Delta t'$ is stationary. So, in particular, for any $\alpha < K$, $(\bigcap t'{}''\alpha)^- = K$; as in 1.1.3, we see that $P = \{\alpha \mid t'(\alpha) = t(\alpha)\}$ is a path through V. Define a continuous non-decreasing unbounded $f: K \rightarrow K$ by $f(\beta) = \bigcup_{\alpha<\beta} V_\alpha$. ($V_\alpha$ is the αth level of V.) Take C, a closed unbounded set of fixed points of f with every element of C a limit ordinal. For each $\beta \in C$, if $\operatorname{rank}_V(\alpha) < \beta$ we have $\alpha < \beta$. ($\operatorname{rank}_V(\alpha)$ is least γ such that $\alpha \in V_\gamma$.) Now $C \cap \Delta t' \subseteq P$, for if $\beta \in C \cap \Delta t'$ then $\forall \alpha \in P$ if $\operatorname{rank}_V(\alpha) < \beta$, then $\beta \in t'(\alpha) = t(\alpha)$, so $\alpha <^* \beta$, i.e., $\beta \in A_\alpha$, so $\beta \in \bigcap_{\alpha \in P} A_\alpha$: as β is a limit, $\beta \in A_\rho$ where ρ is the point in P with $\operatorname{rank}_V(\rho) = \beta$. As $\operatorname{rank}_V(\beta) \leq \beta$, β must be the least point in A_ρ, i.e., $\beta = \rho \in P$.

We now have that P is stationary, since it contains $C \cap \Delta t'$ which is stationary. Now let $h: p \rightarrow 2$ by $h(\alpha) = F(\{\alpha, \beta\})$ for any $\beta \in P$, $\beta > \alpha$. By the construction of T, h is well-defined. As P is stationary and $P = h^{-1}\{0\} \cup h^{-1}\{1\}$, for some j, $h^{-1}\{j\}$ is stationary. Then $h^{-1}\{j\}$ is the desired homogeneous set.

($\Longleftarrow$) Let $t \in S_K$ be given. We need a $t' \sim t$ such that $\Delta t'$ is stationary. If we had available the partition property $K \rightarrow (K)^3$, a simple extention of the argument in Theorem 1.1.3 could be used. We would simply let

$$F(\{\alpha, \beta, \gamma\}_<) = \begin{cases} 0 & \text{if } \{\delta < \alpha \mid \beta \in t(\delta)\} = \{\delta < \alpha \mid \gamma \in t(\delta)\}, \\ 1 & \text{otherwise,} \end{cases}$$

take H, a stationary homogeneous set (so $F''[H]^3 = \{0\}$), then note that for $H' = H \cap \{\zeta \mid \zeta = \bigcup (H \cap \zeta)\}$, H' is also stationary and for $\beta, \gamma \in H'$,

$\{\delta<\beta \mid \beta \in t(\delta)\}=\{\delta<\beta \mid \gamma \in t(\delta)\}$ (since for ordinals $\alpha \in H$, cofinal in β, $\{\delta<\alpha \mid \beta \in t(\delta)\}=\{\delta<\alpha \mid \gamma \in t(\delta)\}$). So defining

$$t'(\alpha)=\begin{cases} t(\alpha) & \text{if every } \beta \in H', \beta>\alpha, \text{ has } \beta \in t(\alpha), \\ K-t(\alpha) & \text{if every } \beta \in H', \beta>\alpha, \text{ has } \beta \notin t(\alpha), \end{cases}$$

we get $H' \subseteq \Delta t'$.

The proof which uses only the property $K \rightarrow_s (K)_2^2$ is more subtle.[8] For each $\alpha<K$ define $r_\alpha: \alpha \rightarrow 2$ by $r_\alpha(\delta)=1$ iff $\alpha \in t(\delta)$. For $r, s: \alpha \rightarrow 2$ say $r<s$ if $r \neq s$ and for $\zeta=$ least δ such that $r(\delta) \neq s(\delta)$ we have $r(\zeta)=0$. $<$ is the usual lexicographical ordering on α-sequences of 0's and 1's. Now let

$$F(\{\alpha, \beta\}_<)=\begin{cases} 0 & \text{if } r_\alpha<r_\beta \restriction \alpha, \\ 1 & \text{otherwise.} \end{cases}$$

Let H be homogeneous for F. Suppose $F''[H]^2=\{0\}$. For each $\alpha \in K$ consider the sequence

$$\{r_\delta \restriction \alpha \mid \delta \in H, \delta>\alpha\}$$

in $\langle {}^\alpha 2, \alpha\rangle$. This sequence is monotone and of length K so there is an $x \in K$, $x \geqslant \alpha$, such that $(\forall \beta, \gamma \in H)(x \leqslant \beta<\gamma \implies r_\beta \restriction \alpha=r_\gamma \restriction \alpha)$. Let $x_\alpha=$ least such x. $x: K \rightarrow K$ is a continuous non-decreasing unbounded function, so we can take C to be a closed unbounded set of fixed points of x. As H is stationary, take $\alpha \in C \cap H$. Since $\alpha \in C$, $x_\alpha=\alpha$, i.e., $(\forall \beta, \gamma \in H)(\alpha \leqslant \beta<\gamma \implies r_\beta \restriction \alpha=r_\gamma \restriction \alpha)$. Since $\alpha \in H$ we can take $\beta=\alpha$ to get $\forall \gamma \in H$ $(\alpha<\gamma \implies r_\alpha=r_\gamma \restriction \alpha)$, a contradiction as $\forall \gamma \in H$ $(\alpha<\gamma \implies r_\alpha<r_\gamma \restriction \alpha)$. So $F''[H]^2=\{1\}$. As above, $\{r_\delta \restriction \alpha \mid \delta \in H, \delta>\alpha\}$ is monotone, so define x_α as above, take C as above, and consider $H'=C \cap H$. We get that

$$(\forall \alpha, \gamma \in H')(\alpha<\gamma \implies r_\alpha=r_\gamma \restriction \alpha).$$

H' is stationary and (using the definition of r_α), for every

$$\alpha, \gamma \in H', \{\delta<\alpha \mid \alpha \in t(\delta)\}=\{\delta<\alpha \mid \gamma \in t(\delta)\},$$

so defining

$$t'(\alpha)=\begin{cases} t(\alpha) & \text{if every } \beta \in H', \beta>\alpha \text{ has } \beta \in t(\alpha) \\ K-t(\alpha) & \text{if every } \beta \in H', \beta>\alpha \text{ has } \beta \notin t(\alpha) \end{cases}$$

we get $H' \subseteq \Delta t'$. $\square$

Remark. There is actually a direct proof of ($\Longleftarrow$) above, avoiding the tree property.[9] Given $F:[K]^2 \rightarrow 2$, let $t(\alpha)=\{\beta>\alpha \mid F(\{\alpha, \beta\})=0\}$. Let $t' \sim t$ be such

[8] This argument, though well-known, is given in detail here since it will be referred to later in 1.4.1.
[9] We thank the referee for providing this direct proof.

that $\Delta t'$ is stationary. $\Delta t' \cap \{\beta \mid t(\beta) = t'(\beta)\}$ and $\Delta t' \cap \{\beta \mid t(\beta) \neq t'(\beta)\}$ are each homogeneous and at least one of them is stationary.

We go to the extra effort of showing that there are stationary paths through trees since getting such paths is a powerful tool which the standard technology of this area can easily manipulate. We will remark on one application in § 1.4.

In summary, then,

Corollary 1.3.5. *The following are equivalent:*

(i) *Given* $A_\alpha \subseteq \alpha$, $\alpha < K$, $(\exists A \subseteq K)$ $(\exists$ *stationary* $H \subseteq K)$ $(\forall \alpha \in H)$ $A \cap \alpha = A_\alpha)$;

(ii) $(\forall t \in S_K)(\exists t')(t' \sim t$ *and* $\Delta t'$ *is stationary*$)$;

(iii) $(\forall \mathscr{C} \subseteq 2^K$ *with* $\bar{\bar{\mathscr{C}}} \leq K)(\exists \mathscr{C}')[\mathscr{C}' \sim \mathscr{C}$ *and* $(\forall u \in S_K)(u''K \subseteq \mathscr{C}' \Longrightarrow (\Delta u)^= = K)]$;

(iv) $K \rightarrow_S (K)^2$.

(i) is the original coherence definition of ineffability;

(ii) is the flipping definition;

(iii) is just a restatement of the "all rearrangements" characterization of Theorem 1.3.2;

(iv) is the partition characterization.

1.4. Extendible flips which give stationary diagonal intersections

One obvious way to add uniformity to obtain a stronger definition would be to take an ultra-uniform version of (iii) namely, $(\exists \mathscr{C}')(\mathscr{C}' \sim 2^K)$ and $(\forall u \in S_K)$ $(u''K \subseteq \mathscr{C}' \Longrightarrow (\Delta u)^= = K)$, that is, to assert that one can flip *all* subsets of K simultaneously so that small collections of subsets have large diagonal intersection. It is easily seen that $\mathscr{C}'$ is such a flip iff $\mathscr{C}'$ is a normal K-additive ultrafilter on K, i.e., a normal measure on K. A less uniform property than measurability might be one which allows an iteration of the flipping property, i.e., given K-many subsets of K we can flip them so that diagonal intersections of sets drawn from the resulting collection are large and such that given such a flip we can then flip another K-size collection of subsets, so that diagonal intersections of sets drawn from the union of the two collections are still large. In the cases of strongly inaccessible or weakly compact cardinals, this type of iterability was automatic, simply because a K-size subset of a K-size subset of K is still a K-size subset of K. But if $H = \{h_\delta \mid \delta < K\}$ is a stationary subset of K and H' is stationary, $\{h_\delta \mid \delta \in H'\}$ need not be a stationary subset of K. We will see later that adding the ability to iterate does give a stronger hypothesis than ineffability, namely "complete ineffability."

First, note that by Lemma 1.3.3, if $\mathscr{C} \subseteq 2^K$, $\bar{\bar{\mathscr{C}}} \leq K$ and $t, t' \in S_K$ are such that $t''K = t'''K = \mathscr{C}$, then there is a closed unbounded C such that $C \cap \Delta t = C \cap \Delta t'$, so that modulo complements of closed unbounded sets we may speak of *the* diagonal intersection of $\mathscr{C}$. Formally, let $\mathbf{0} = \{N \subseteq K \mid K - N$ contains a closed unbounded set}. $\mathbf{0}$ is a K-additive ideal in 2^K whenever K is a regular cardinal, in which case we define $\Delta(\mathscr{C})$ = the equivalence class of Δt in $2^K/\mathbf{0}$ for any t such that $t''K = \mathscr{C}$.

Definition. K is completely *ineffable* iff K is regular and $\exists Q \subseteq 2^{2^K}$ such that

(i) $(\forall \mathscr{C} \in Q)\bar{\bar{\mathscr{C}}} \leq K$ and $\Delta\mathscr{C} \neq \mathbf{0}$

(ii) $(\forall \mathscr{C} \subseteq 2^K$ with $\bar{\bar{\mathscr{C}}} \leq K)(\exists \mathscr{C}' \in Q)(\mathscr{C}' \sim \mathscr{C})$

(iii) $(\forall \mathscr{C} \in Q)(\forall \mathscr{D} \subseteq 2^K$ with $\bar{\bar{\mathscr{D}}} \leq K)$ $(\exists \mathscr{D}')[\mathscr{D}' \sim \mathscr{D}$ and $\mathscr{C} \cup \mathscr{D}' \in Q]$.

Q is thought of as a collection of excellent flips. (i) and (ii) say thatb every small collection of subsets has a flip in Q with stationary diagonal intersection. (iii) says that a flip in Q can be extended to include K-many more subsets.

It is easy to check that a measurable cardinal, K, with normal measure μ is completely ineffable, with

$$Q = \{\mathscr{C} \in 2^{2^K} \mid \bar{\bar{\mathscr{C}}} \leq K \text{ and } (\forall A \in \mathscr{C})\mu(A) = 1\}.$$

A completely ineffable cardinal is clearly ineffable. We will see in the section on indescribability properties that complete ineffability is stronger than ineffability.

Here is a convenient

Definition. $R \subseteq 2^K$ is a *stationary class* if

(i) $R \neq \emptyset$

(ii) $A \in R \implies A$ is stationary

(iii) $(A \in R$ and $B \supseteq A) \implies B \in R$.

Using the techniques developed above, we can easily show

Theorem 1.4. *The following are equivalent*

(i) *K is completely ineffable*

(ii) *$\exists$ a stationary class R such that given $H \in R$ and a sequence $\{A_\alpha \mid \alpha < K\}$ with each $A_\alpha \subseteq \alpha$, there is an $H' \subseteq H$, $H' \in R$ and an $A \subseteq K$ such that $(\forall \alpha \in H')[A \cap \alpha = A_\alpha]$.*

(iii) *$\exists$ a stationary class R such that given $H \in R$ and $t \in S_K$ there is a t' and an $H' \subseteq H$ such that $t' \sim t$, $H' \in R$, and $\Delta t' \supseteq H'$.*

(iv) *$\exists$ a stationary class R such that given $H \in R$ and $F: [H]^2 \to 2$ there is an $H' \subseteq H$, $H' \in R$, H' homogeneous for F.*

(i) says there is a collection of excellent flips, (ii) is a strengthened coherence property, (iii) is a strengthened diagonal intersection-type flipping property, (iv) says there is a class closed under finding a homogeneous set for $K \to (K)^2$ type partitions.

Proof. ((i) $\implies$ (ii)): Let Q be as in the definition of completely ineffable. Take

$$R = \{A \mid A \supseteq \Delta(t) \quad \text{for some } t \in S_K \text{ such that } t''K \in Q\}.$$

R is clearly a stationary class. Now, given $H \in R$ and $\{A_\alpha \mid \alpha < K\}$, $A_\alpha \subseteq \alpha$, let $\mathscr{C} \in Q$ and let $c \in S_K$ be such that $c''K = \mathscr{C}$ and $\Delta c \subseteq H$. Let $d(\alpha) = \{\beta < K \mid \alpha \in A_\beta\}$, and $\mathscr{D} = d''K$. Find $\mathscr{D}' \sim \mathscr{D}$ such that $\mathscr{C} \cup \mathscr{D}' \in Q$. Let $d' \sim d$ be such that $d'''K = \mathscr{D}'$.

Define $e \in S_K$ by

$$e(\lambda + n) = \begin{cases} c\left(\lambda + \frac{n}{2}\right) & \text{for } \lambda = 0 \text{ or a limit, } n < \omega,\ n \text{ even} \\ d'\left(\lambda + \frac{n-1}{2}\right) & \text{for } \lambda = 0 \text{ or a limit, } n < \omega,\ n \text{ odd} \end{cases}$$

$e''K = \mathscr{C} \cup \mathscr{D}'$ and it is easily seen that for $\mathrm{Lim} = \{\lambda < K \mid \lambda \text{ a limit}\}$, $\Delta e \cap \mathrm{Lim} \subseteq \Delta c$ and, as in the proof of Theorem 1.2.1, for $A = \bigcup_{\alpha \in \Delta e} A_\alpha$, we have $(\forall \alpha \in \Delta e)[A \cap \alpha = A_\alpha]$. We need only check that for $H' = \Delta e \cap \mathrm{Lim}$ we have $H' \in R$, which is immediate from the following.

Lemma 1.4.2. *Suppose $e \in S_K$ is such, that $e''K \in Q$ and C is a closed unbounded subset of K. Then $(\exists t \in S_K)$ such that $t''K \in Q$ and $\Delta t \subseteq \Delta e \cap C$.*

Proof of Lemma. Let $\mathscr{J} = \{C\} \cup \{K - \alpha \mid \alpha < K\}$. Find $\mathscr{J}' \sim \mathscr{J}$ such that $t''K \cup \mathscr{J}' \in Q$. We know that for any u with $u''K \subseteq t''K \cup \mathscr{J}'$, Δu is stationary, so clearly $\mathscr{J}' = \mathscr{J}$. Consider $t \in S_K$ defined by $t(0) = K - \omega$, $t(1) = C$, $t(n+2) = e(n)$ for $n < \omega$, $t(\xi) = e(\xi)$ for $\xi \geq \omega$. $\Delta t \subseteq C$ and $\Delta t \subseteq \Delta e$, so $\Delta t \subseteq \Delta e \cap C$. □

((ii)→(iii)) Let R be as in (ii). We show that R satisfies the requirements of (iii). Let $H \in R$ and $t \in S_K$ be given. Let $A_\alpha = \{\beta < \alpha \mid \alpha \in t(\beta)\}$. Take the H' and A provided by (ii). Define

$$t'(\beta) = \begin{cases} t(\beta) & \text{if } \beta \in A \\ K - t(\beta) & \text{if } \beta \notin A \end{cases}$$

as in Theorem 1.2.1, $\Delta t' \supseteq H'$.

((iii)→(iv)) Let R be as in (iii). We show that R is as required in (iv). Let $H \in R$ and $F: [H]^2 \to 2$ be given. Build trees T and V as in the proof of Theorem 1.3.4, except that in this case the nodes in T are subsets of H, and V is of the form $\langle H \leqslant \rangle$. Applying property (iii) to the argument of Theorem 1.3.4 we get a path, P, through V, a closed unbounded set C, and an $H' \subseteq H$, $H' \in R$, such that $H' \cap C \subseteq P$. For $i \in \{0, 1\}$, let

$$H'_i = \{\alpha \in H' \cap C \mid (\exists \beta \in P)[\beta \geq \alpha \text{ and } F(\{\alpha, \beta\}) = i]\}.$$

By the construction of T, each H'_i is homogeneous for F and $H'_0 \cup H'_1 = H' \cap C$. We need only show that some $H'_i \in R$. It is easy to check that for any $J \in R$ and closed unbounded D, $J \cap D \in R$. (Just apply (iii) to t defined by $(\forall \alpha < K)\, t(\alpha) = D$.) So $H' \cap C \in R$. Consider $t \in S_K$ defined by $t(0) = K - \omega$, $t(1) = H_0'$, $t(2) = H_1'$, and $t(\xi) = K$ for $\xi > 2$. By (iii), take $t' \sim t$ and $H'' \in R$, $H'' \subseteq H' \cap C$, such that $\Delta t' \supseteq H''$. If $t'(1) = K - H_0'$ and $t'(2) = K - H_1'$, then $\Delta t' \cap (H' \cap C) = \emptyset$, so for some $j \in \{0, 1\}$, $t'(1 + j) = H_j'$. So $H'' \subseteq \Delta t' \subseteq H_j'$, so $H_j' \in R$ and H_j' is the desired homogeneous set.

((iv)$\to$(i)) Let

$$Q = \{\mathscr{C} \subseteq 2^K \mid \bar{\bar{\mathscr{C}}} \leq K \text{ and } (\exists t \in S_K)[t''K = \mathscr{C} \text{ and } \Delta t \in R]\}.$$

Q evidently satisfies (i) of the definition of complete ineffability. As for (ii) of the definition, given $\mathscr{C} \subseteq 2^K$, $\bar{\bar{\mathscr{C}}} \leq K$, let t be such that $t''K = \mathscr{C}$ and let $H \in R$ be homogeneous for the partition of Theorem 1.3.4. Then, as we saw before, we get a closed unbounded set C and a $t' \sim t$ such that $C \cap H \subseteq \Delta t'$. It is easy to see that $C \cap H \in R$. (Consider the partition of $[H]^2$ given by $\{\alpha, \beta\}_< \mapsto 0$ iff $\alpha \in C$.) So $\mathscr{C}' = t'''K$ is as desired.

As for (iii), let $\mathscr{C} \in Q$, $\mathscr{D} \subseteq 2^K$, $\bar{\bar{\mathscr{D}}} \leq K$. Take c such that $c''K = \mathscr{C}$ and $\Delta c \in R$. Choose d such that $d''K = \mathscr{D}$. Let $H = \Delta c$.

Now, define $t \in S_K$ by

$$t(\lambda + n) = \begin{cases} c\left(\lambda + \dfrac{n}{2}\right) & \text{for } \lambda = 0 \text{ or a limit, } n \text{ an even integer} \\[2ex] d\left(\lambda + \dfrac{n-1}{2}\right) & \text{for } \lambda = 0 \text{ or a limit, } n \text{ an odd integer.} \end{cases}$$

For $\beta \in H$ let $r_\beta : \beta \to 2$ by $r_\beta(\delta) = 1$ iff $\beta \in t(\delta)$. Let $F : [H]^2 \to 2$ by

$$F(\{\alpha, \beta\}_<) = \begin{cases} 0 & \text{if } r_\alpha < r_\beta \restriction \alpha \\[1ex] 1 & \text{otherwise.} \end{cases}$$

Take $H' \subseteq H$, $H' \in R$, H' homogeneous for F as in Theorem 1.3.4, $F''[H]^2 = \{1\}$ and there is a closed set, G, and a $t' \sim t$ such that $G \cap H' \subseteq \Delta t'$. As above, $H'' = G \cap H' \in R$, so $\Delta t' \in R$.

Now, suppose for some $\lambda + n$, $\lambda = 0$ or a limit, n an even integer, we have $t'(\lambda + n) \neq t(\lambda + n)$. Then

$$(\Delta t' \cap (K - (\lambda + \omega))) \cap c\left(\lambda + \frac{n}{2}\right) = \emptyset.$$

But $H - (\lambda + \omega) \subseteq c\left(\lambda + \dfrac{n}{2}\right)$ and $H'' \subseteq H$, so $(H'' - (\lambda + \omega)) \cap \Delta t' = \emptyset$. $H'' \subseteq \Delta t'$ so we have $H'' \subseteq \lambda + \omega$ contradicting that H'' is stationary. Thus, letting

$$\mathscr{D}' = t''' \{\lambda + n \mid \lambda = 0 \text{ or a limit, } n \text{ an odd integer}\},$$

we have $\mathscr{D}' \sim \mathscr{D}$ and $t'''K = \mathscr{C} \cup \mathscr{D}'$, so $\mathscr{C} \cup \mathscr{D}' \in Q$. $\square$

Remark. The proof above of ((iii) $\rightarrow$ (iv)) actually gives the tree property with a path in R. This, combined with the standard inductive proof showing $\kappa \rightarrow (\kappa)^2 \Rightarrow \kappa \rightarrow (\kappa)^n$, easily gives, for each $n \in \omega$, that R is closed under finding homogeneous subsets for $\kappa \rightarrow (\kappa)^n$ type partitions.

We are now armed with sufficient combinatorial power to study model theoretic properties of these cardinals, such as indescribability and existence of outside ultrafilters.

2. Construction of outside ultrafilters by forcing

Successively stronger flipping properties can obviously be regarded as successive approximations to measurability. One goes from one property to a stronger one by adding more of a measure's inherent uniformity. But, in fact, these cardinals can actually be thought of as *having* measures — only these measures are on the "outside", rather than actually existing in the given model of set theory. Roughly speaking, one gets from a strongly inaccessible to a weakly compact by demanding more of the measure to be inside the model, and thence to a completely ineffable by demanding that the measure be "normal". Let us be more precise now.

Throughout this section, M is a countable standard, transitive model of ZFC, and K is an uncountable cardinal of M. $\mathcal{U} \subseteq (2^K)^M$ is called an M-ultrafilter iff

(i) $(\forall \alpha < K)[\{\alpha\} \notin \mathcal{U}]$ and $\emptyset \notin \mathcal{U}$.

(ii) For $A \subseteq K$, $A \in M$, either $A \in \mathcal{U}$ or $K - A \in \mathcal{U}$.

(iii) If the sequence $\{A_\alpha \mid \alpha < \delta < K\}$ is in M, and each $A_\alpha \in \mathcal{U}$, then $\bigcap_{\alpha<\delta} A_\alpha \in \mathcal{U}$.

(iv) If the sequence $\{A_\alpha \mid \alpha < K\}$ is in M, then $\{\alpha < K \mid A_\alpha \in \mathcal{U}\}$ is also in M.

$\mathcal{U}$ may be outside of M i.e., not a member of M, but clause (iv) says that small pieces of $\mathcal{U}$ are in M. If (iv) is weakened to

(iv′) If the sequence $\{A_\alpha \mid \alpha < \delta < K\}$ is in M, then $\{\alpha < \delta \mid A_\alpha \in \mathcal{U}\}$ is also in M,

then $\mathcal{U}$ is called a far-outside-M-ultrafilter. If $\mathcal{U}$ is an M-ultrafilter, we say $\mathcal{U}$ is normal if

(v) Whenever $\{A_\alpha \mid \alpha < K\}$ is in M, and each $A_\alpha \in \mathcal{U}$, then $\Delta\{A_\alpha \mid \alpha \in K\} \in \mathcal{U}$.

Normal M-ultrafilters have been studied by Kunen [22], who uses them to take iterated ultraproducts of M.

A normal M-ultrafilter gives us some of the power of an actual normal measure, enough to do an argument similar to the one giving a strong indescribability result for measurable cardinals. But, of course, the indescribability of these cardinals is less strong than that of a measurable. In the next section we exactly characterize this strength. Right now we will show that certain combinatorial properties of K in M are equivalent to K carrying certain M-ultrafilters.

2.1. Non-normal M-ultrafilters

Theorem 2.1.1. *$(\exists \mathcal{U})[\mathcal{U}$ is a far-outside-M-ultrafilter on $K]$ iff $M \models K$ is strongly inaccessible.*

Proof. ($\Longleftarrow$) We force with the partial ordering

$$\mathcal{P} = \langle\{\mathscr{C} \subsetneq 2^{K} \mid \bar{\bar{\mathscr{C}}} < K \text{ and } (\bigcap \mathscr{C})^{-} = K\}, \subsetneq\rangle.$$

Let G be an M-generic subset of $\mathcal{P}$. Let $\mathcal{U} = \bigcup G$. We use standard density arguments to show that $\mathcal{U}$ satisfies (i), (ii), (iii) and (iv′).

$\mathcal{U}$ evidently satisfies (i), since for no $\alpha < K$ and $p \in \mathcal{P}$ do we have $\{\alpha\} \in p$, and no $p \in \mathcal{P}$ has $\emptyset \in p$.

Given $A \subsetneq K$ and $p \in \mathcal{P}$, by Corollary 1.1.2, either $p \cup \{A\} \in \mathcal{P}$ or $p \cup \{K - A\} \in \mathcal{P}$. So

$$\{p \mid p \Vdash A \in \mathcal{U} \text{ or } p \Vdash K - A \in \mathcal{U}\}$$

is dense. So $\mathcal{U}$ satisfies (ii).

Now, for (iii), let $\{A_\alpha \mid \alpha < \delta < K\}$ be in M and suppose each $A_\alpha \in \mathcal{U}$. By Corollary 1.1.2,

$$D = \{q \mid (\forall \alpha < \delta)(A_\alpha \in q \text{ or } K - A_\alpha \in q)\}$$

is dense. Let $q \in D \cap G$. If for some α, $K - A_\alpha \in q$, then since some $p \in G$ has $A_\alpha \in p$, we would have that for any $q' \in G$, with $q' \supsetneq q$ and $q' \supsetneq p$ both A_α and $K - A_\alpha$ are in q'. So for each α, $A_\alpha \in q$. By Corollary 1.1.2 again, there is an $r \supsetneq q$, $r \in G$ such that either $\bigcap_{\alpha<\delta} A_\alpha \in r$ or $K - \bigcap_{\alpha<\delta} A_\alpha \in r$. But if the second held $\bigcap r = \emptyset$. Thus $\bigcap_{\alpha<\delta} A_\alpha \in r$ and thus $\in \mathcal{U}$.

As for (iv′), given $\{A_\alpha \mid \alpha < \delta < K\}$, take $q \in G$ such that $(\forall \alpha)[A_\alpha \in q$ or $K - A_\alpha \in q]$. Then it is easy to see that $A_\alpha \in \mathcal{U} \Longleftrightarrow A_\alpha \in q$, so $\{\alpha < \delta \mid A_\alpha \in \mathcal{U}\} \in M$.

($\Longrightarrow$) We prove the flipping characterization of strong inaccessibility holds for K. Given $t \in s_K$ let

$$t'(\alpha) = \begin{cases} t(\alpha) & \text{if } \alpha \in \{\alpha < \delta \mid A_\alpha \in \mathcal{U}\} \\ K - t(\alpha) & \text{otherwise} \end{cases}$$

$t' \sim t$, by (iv′) $t'(\alpha) \in M$, and by (ii)+(iii), $\bigcap t' \in \mathcal{U}$, so $(\bigcap t')^{-} = K$, since by (i), (ii), and (iii), no element of $\mathcal{U}$ can have size less than K. $\square$

Theorem 2.1.2 (Kunen [21]). *$(\exists \mathcal{U})[\mathcal{U}$ is an M-ultrafilter on $K] \Longleftrightarrow M \vDash K$ is weakly compact.*

Proof. ($\Longleftarrow$) Let

$$\mathcal{P} = \langle\{\mathscr{C} \leq 2^{K} \mid \bar{\bar{\mathscr{C}}} \leq K \text{ and } (\forall \mathscr{E} \subsetneq C \text{ with } \bar{\bar{\mathscr{E}}} < K)[(\bigcap \mathscr{E})^{-} = K]\}, \subsetneq\rangle.$$

Let G be an M-generic subset of $\mathcal{P}$, and let $\mathcal{U} = \bigcup G$. Using Corollary 1.1.4, the same arguments used above show that $\mathcal{U}$ satisfies (i), (ii), and (iii). Since by Corollary 1.1.4, given

$$\{A_\alpha \mid \alpha < K\} \in M,\ D = \{p \mid (\forall \alpha < K)[A_\alpha \in p \text{ or } K - A_\alpha \in p]\}$$

is dense, for any $p \in D \cap G$ we have $A_\alpha \in \mathcal{U} \iff A_\alpha \in p$, so $\mathcal{U}$ also satisfies (iv). In short, K-length pieces of $\mathcal{U}$ are in M since the flipping property for weakly compacts allows us to use conditions which can always be extended to decide about K-many more sets. For strongly inaccessible cardinals, our conditions could only be extended to handle less-than-K-many more sets, so, correspondingly, only less-than-K-size pieces of $\mathcal{U}$ can be expected to be in M.

($\Longrightarrow$) Given $t \in S_K$, let

$$t'(\alpha) = \begin{cases} t(\alpha) & \text{if } \alpha \in \{\alpha < K \mid t(\alpha) \in \mathcal{U}\} \\ K - t(\alpha) & \text{otherwise} \end{cases}$$

$t' \sim t$ and by (iv) $t' \in M$, so by (ii) and (iii), for any $\delta < K$, $\bigcap t'{}''\delta \in \mathcal{U}$, so $(\bigcap t'{}''\delta)^{-} = K$.

2.2. Normal M-ultrafilters

Theorem 2.2. (Kleinberg [19]). $(\exists \mathcal{U})[\mathcal{U}$ *is a normal M-ultrafilter on* $K] \iff M \models K$ *is completely ineffable.*

Proof. ($\Longleftarrow$) Take Q as in the definition of completely ineffable. Force with the partial ordering $\langle Q, \subseteq \rangle$. Let G be an M-generic subset of Q, $\mathcal{U} = \bigcup G$. The properties of Q give immediately the density arguments necessary to show $\mathcal{U}$ an M-ultrafilter. For normality, suppose $\{A_\alpha \mid \alpha < K\} \in M$ and each $A_\alpha \in \mathcal{U}$. Then by usual density arguments, there is a condition $p \in Q \cap G$ such that $\{A_\alpha \mid \alpha < K\} \subseteq p$. Now if $X = \Delta\{A_\alpha \mid \alpha < K\} \notin \mathcal{U}$, then $K - X \in \mathcal{U}$. So there is a $p' \in G$, $p \subseteq p'$ such that $K - X \in p'$. Choose $t \in S_K$ such that $t''K = p'$ and $t(0) = K - X$ and let $v \in S_K$ be defined by $v(\alpha) = A_\alpha$. We have $v''K \subseteq t''K$, so by Lemma 1.1.3, there is a closed unbounded C such that $\Delta v \supseteq C \cap \Delta t$. Now $X = \Delta v$, so $X \supseteq C \cap \Delta t$, so $X \cap C \cap \Delta t = C \cap \Delta t$. But $X \cap \Delta t = \emptyset$, so $C \cap \Delta t = \emptyset$, $\Delta p' = \mathbf{0}$, a contradiction.

($\Longrightarrow$) We are given a normal M-ultrafilter on K, $\mathcal{U}$. Showing that K is completely ineffable in M will be far more difficult than the corresponding problem for strongly inaccessibles or weakly compacts, since we are required to construct a collection of some sort, rather than to prove a more local property.

In M there is an obvious way to go about constructing a Q as in the definition of completely ineffable. Let

$$Q_0 = \{\mathscr{C} \subseteq 2^{2^K} \mid \bar{\bar{\mathscr{C}}} \leq K \text{ and } \Delta\mathscr{C} \neq \mathbf{0}\},$$

$$Q_{\alpha+1} = \{\mathscr{C} \in Q_\alpha \mid (\forall \mathscr{D} \subseteq 2^K \text{ with } \bar{\bar{\mathscr{D}}} \leq K)(\exists \mathscr{D}')[\mathscr{D}' \sim \mathscr{D} \text{ and } \mathscr{C} \cup \mathscr{D}' \in Q_\alpha]\}$$

$$Q_\lambda = \bigcap_{\beta < \lambda} Q_\beta \quad \text{for } \lambda \text{ a limit.}$$

Then

$$\{Q_\eta\}_{\eta\in\theta} \text{ is a decreasing sequence (i.e. } Q_\beta \subseteq Q_\alpha \quad \text{for } \beta > \alpha),$$

so for some ordinal θ, $Q_\theta = Q_\delta$ for all $\delta \geq \theta$.

Now, notice that by 1.3.3, it is clear (by induction) that each Q_α is closed under subcollection, i.e., if $\mathscr{C} \in Q_\alpha$ and $\mathscr{D} \subseteq \mathscr{C}$, then $\mathscr{D} \in Q_\alpha$. Of course, Q_θ satisfies (i) of the definition of complete ineffability, and since $Q_\theta = Q_{\theta+1}$, Q_θ satisfies (iii). If $Q_\theta \neq \emptyset$, then its satisfaction of (iii) implies its satisfaction of (ii), since Q_θ is closed under subcollection.

Now is $Q_\theta \neq \emptyset$? Well, first notice that the usual proof that a normal measure assigns measure 1 to any closed unbounded set goes through for $\mathscr{U}$, so every element of $\mathscr{U}$ is stationary. Thus, letting

$$\hat{\mathscr{U}} = \{\mathscr{C} \subseteq 2^K \mid \mathscr{C} \in M, \bar{\bar{\mathscr{C}}}^M \leq K, \text{ and } \mathscr{C} \subseteq \mathscr{U}\}, \hat{\mathscr{U}} \subseteq Q_0.$$

Now by induction we easily have $\hat{\mathscr{U}} \subseteq Q_\alpha$ for every α. So $\hat{\mathscr{U}} \subseteq Q_\theta \neq \emptyset$. Thus $M \models K$ is completely ineffable. □

It will be important later to have a better handle on the ordinal θ above. Let K be any regular cardinal of M, and let $\{Q_\alpha\}_{\alpha \text{ an ordinal}}$ be as above. Each $\mathscr{C} \in Q_0$ can be viewed (coded) as a subset of K, so Q_0 can be viewed as $\subseteq 2^K$. Thus for θ the least ordinal δ such that $Q_{\delta+1} = Q_\delta$, $\theta < (2^K)^+$. In fact, let $\mathfrak{A}$ = smallest (standard transitive) admissible set with $R(K+1)$ as an element. ($\mathfrak{A}$ is of the form $L_\gamma(\langle A\rangle_{A\in R(K)})$.) Then, we have

Fact 2.2. $\theta \leq \gamma$.

Proof. Say not. Then let $\mathscr{C} \in Q_\gamma - Q_{\gamma+1}$. Take any $\mathscr{D}$ which caused $\mathscr{C}$ to be thrown out at stage γ, i.e. such that for no $\mathscr{D}' \sim \mathscr{D}$ was $\mathscr{C} \cup \mathscr{D}' \in Q_\gamma$. Let

$$X = \{\mathscr{D}' \mid \mathscr{D}' \sim \mathscr{D} \quad \text{and} \quad \mathscr{C} \cup \mathscr{D}' \in Q_0\}\ X \in \mathfrak{A},$$

and in $\mathfrak{A}$ we can define $m: X \to \gamma$ by $m(\mathscr{D}') =$ least δ, such that $\mathscr{C} \cup \mathscr{D}' \notin Q_\delta$. By admissibility, $\bigcup_{\mathscr{D}'\in X} m(\mathscr{D}') < \gamma$, contradicting that $\mathscr{C} \in Q_\gamma$. □

We will see later that for K completely ineffable, θ actually equals γ.

Fact 2.2. If K is not completely ineffable then for some $\delta < \gamma$, $Q_\delta = \emptyset$.

Proof. $Q_\theta = \emptyset$, else K would be completely ineffable. So by Fact 2.2.2, $Q_\gamma = \emptyset$. Then defining $m: Q_0 \to \gamma$ by $m(\mathscr{C}) =$ least δ such that $\mathscr{C} \notin Q_\delta$, the admissibility of $\mathfrak{A}$ shows that for $\delta = \bigcup_{\mathscr{C}\in Q_0} m(\mathscr{C})$, we have $\delta < \gamma$, and clearly $Q_\delta = \emptyset$. □

Corollary 2.2.4. *K is completely ineffable iff*

$$(\forall \alpha < \gamma)(Q_\alpha \neq \emptyset).$$

Proof. ($\Longrightarrow$) If Q demonstrates the complete ineffability of K, then for all α, $Q \subseteq Q_\alpha$.

($\Longleftarrow$) Immediate from 2.2.3. $\square$

Remark. One natural example of a completely ineffable arises as follows: Suppose $\delta \to (\omega)^{<\omega}$. Let $I = \{i_0, i_1, \ldots\}$ be an ω-sequence of indiscernibles for $R(\delta)$, and H = Skolem hull of I in $R(\delta)$. Then the map $i_n \mapsto i_{n+1}$ induces an elementary embedding of H into itself. One shows that if K is the least ordinal moved by this embedding then there is a normal H-ultrafilter on K. So $H \models K$ is completely ineffable. As $H < R(\delta)$, K really is completely ineffable. (For details, see Kleinberg [19].)

In the next section we use our combinatorial knowledge about the cardinals we have studied and the existence of outside ultrafilters to prove indescribability results. We will show that our successively stronger flipping properties actually give rise to successively stronger larger cardinal hypotheses.

3. Ultraproduct and indescribability

3.1. *Introduction and definitions*

In this section we will compare the sizes of the various cardinals discussed. Although many of the theorems of this section are well known[10] we will present proofs using weak ultraproducts inspired by flipping properties, designed to lead up to a precise characterization of the indescribability strength of completely ineffables. This information will yield some interesting remarks about the forcing done in the previous section.

Immediately from the flipping definitions comes the following chain of implications: K is measurable $\Longrightarrow$ K is completely ineffable $\Longrightarrow$ K is ineffable $\Longrightarrow$ K is weakly ineffable $\Longrightarrow$ K is weakly compact $\Longrightarrow$ K is strongly inaccessible. A more useful comparison would show that beneath (less than) any type B cardinal there lies a type A cardinal. Such a result is fraught with the usual connotations: If K is the smallest type B then $R(K) \models$ ZFC + "there exist type A's, but no type B's", so the axiom of infinity, "there exists a type B", is strictly stronger than "$\exists$ a type A". Also, since "$\exists$ type B" implies the consistency of ZFC + "$\exists$ type A", Gödel's second theorem tells us that there is no relative consistency proof — if ZFC + $\exists$ type A is consistent, then ZFC + "Consistency (ZFC + $\exists$ type A)" $\nvdash$ "Consistency (ZFC + $\exists$ type B)".

The most widely used approach in constructing such a comparison involves the use of indescribability (or reflectivity) results. A cardinal K is said to be Σ-indescribable (where Σ is a class of sentences) if for every $\varphi \in \Sigma$, if φ is true in $R(K)$, then it is true in $R(\alpha)$ for some $\alpha < K$. Thus, if we can show that the

[10] See Hanf–Scott [11.5] and Jensen–Kunen [13].

sentence "I am a type A cardinal" is an element of the indescribability class for type B's and if all type B's are type A's, then there is a type A beneath every type B. This sort of attack works for all links in our chain but one. The weighing of weakly compacts versus weakly ineffables will require a different treatment. Let the reader be warned against viewing a cardinal's reflecting class as a direct measure of its size. As we shall show, beneath any weakly ineffable (itself only π_2^1-indescribable) there is a K which is π_m^n indescribable for all m and n.

Definition. By π_m^n we mean the *collection of formulas* φ, in the language of ZF, such that

(1) φ is in prenex normal form, with higher-order quantifiers preceding lower-order ones,

(2) φ has at most m alternations of type-n quantifiers, and has no higher-than-n-type quantifiers,

(3) if φ has fully m-many type-n quantifiers, the first one is a universal quantifier,

(4) φ may have several constant symbols,

(5) φ may also have a special, one-place predicate symbol X.

Σ_m^n is defined similarly, with "existential" replacing "universal" in clause (3). Type n quantifiers are interpreted as ranging over elements of $R(K+n)$ and the one-place predicate symbol X has as interpretation some subset, X, of $R(K)$, of our choice; X is referred to as a parameter. The constant symbols are interpreted as elements of $R(K)$ of our choice.

Since in most cases of interest $(R(K))^{-} = \bar{\bar{K}}$, we often think of parameters as being subsets of K; type 1 quantifiers may be considered to range over subsets of K.

The use of parameters is of key importance in the applications of indescribability. Nevertheless, we shall frequently supress mention of them in the proofs, as when they can be "carried along" without special allowance.

Definition[11]**.** For φ a sentence in π_m^n, K some ordinal and $X \subseteq R(K)$, we say that φ *reflects* over $R(K)$ (or "K reflects φ"), when

$$(\langle R(K), X, \varepsilon\rangle \models \varphi) \Longrightarrow (\exists \alpha < K)(\langle R(\alpha), X \cap R(\alpha), \varepsilon\rangle \models \varphi).$$

3.2. Measurables

Our first theorem will be the well-known result that measurable cardinals are π_1^2-indescribable. We would like to think of flipping properties as weakenings of measurability. The indescribability proofs for the other cardinals will therefore be presented as modified versions of the ultraproduct proof for measurables.

[11] Note that if K is π_n^1 reflecting, it is automatically Σ_{n+1}^1 reflecting, for if $R(K) \models \exists Y \psi(Y)$, then $R(K) \models \psi(Y_0)$ for some $Y_0 \subseteq R(K)$. Now $\psi(Y_0)$ is π_n^1 in an additional parameter, so $\langle R(\alpha), \varepsilon, Y_0 \cap R(\alpha)\rangle \models \psi(Y_0)$, so $\langle R(\alpha), \varepsilon\rangle \models \exists Y \psi(Y)$.

Lemma 3.2.1. *Let K be a measurable cardinal, and μ be a normal measure on K. Then $\prod_{\alpha<K}\langle R(\alpha),\varepsilon\rangle/\mu \cong \langle R(K),\varepsilon\rangle$.*

Proof. Claim. For any $[f]$[12] $\in \prod_{\alpha<K} R(\alpha)/\mu$, there is an $x_0 \in R(K)$, such that $\mu(\{\alpha \mid f(\alpha)=x_0\})=1$. (*Proof of claim*: Let ρ be the function that assigns the ordinal rank of a set; then $\rho\circ f\colon K\to K$ and $\rho\circ f(\alpha)<\alpha$ for all α. By normality of μ, then, there is a single β such that $(\rho\circ f)(\alpha)=\beta$ almost everywhere, so $f(\alpha)\in R(\beta)$ almost everywhere. For each $x\in R(\beta)$ set $f_x=\{\alpha \mid f(\alpha)=x\}$, then $\mu(\bigcup_{x\in R(\beta)} f_x)=1$, so since $(R(\beta))^{=}<K$ and μ is K-additive, $\mu(f_{x_0})=1$ for some x_0.)

If we let

$$e_0\colon \prod_{\alpha<K} R(\alpha)/\mu \to R(K)$$

by $e_0([f])=$ the x such that $\mu(\{\alpha \mid f(\alpha)=x\})=1$, then it is easy to check e_0 is a bijection and an isomorphism of the corresponding structures. □

Lemma 3.2.2. *Let K, μ be as above, then $\prod_{\alpha<K}\langle R(\alpha+1),\varepsilon\rangle/\mu \cong \langle R(K+1),\varepsilon\rangle$.*

Proof. Let $e_1\colon \prod_{\alpha<K} R(\alpha+1)/\mu \to R(K+1)$ by

$$e_1([f])=\{x\in R(K) \mid \mu(\{\alpha \mid x\in f(\alpha)\})=1\}.$$

Then e_1 is one-one (refer to previous lemma when checking) and onto, for if $X\subseteq R(K)$ set $f(\alpha)=R(\alpha)\cap X$. Then $e_1([f])=X$. It is easy to see e_1 is an isomorphism between the corresponding structures. □

Lemma 3.2.3. *With K, μ as before, there is a* 1:1 *map*

$$e_2\colon \prod_{\alpha<K}\langle R(\alpha+2),\varepsilon\rangle/\mu \to \langle R(K+2),\varepsilon\rangle.$$

e_2 is an extension of e_1 and an embedding of structures, i.e., if

$$e_2\left(\left\{A_\alpha\right\}_{\alpha<K}\right)=A\in R(K+2)$$

then

$$e_1\colon \prod_{\alpha<K}\langle R(\alpha+1),\varepsilon,A_\alpha\rangle/\mu \to \langle R(K+1),\varepsilon,A\rangle$$

is an isomorphism.

N.B. e_2 is neither onto (μ is not in its range) nor elementary.

Proof. Let $A_\alpha\in R(\alpha+2)$ for each $\alpha<K$. Then set

[12] By $[f]$ we mean the equivalence class, modulo μ, of $f\in\prod_{\alpha<K}R(\alpha)$. For background on measures and ultraproducts see Kleinberg [17].

$$e_2(\{A_\alpha\}) = \{x \in R(K+1) \mid \mu(\{\alpha \mid e_1^{-1}(x)(\alpha) \in A_\alpha\}) = 1\} \quad \square.$$

Theorem 3.2.4. *A measurable cardinal is π_1^2-indescribable.*

Proof. Let $(\forall X)(\psi(X))$ be a π_1^2-sentence true over $\langle R(K+2), \varepsilon\rangle$, where $\psi(X)$ is π_n^1 for some n. Say $(\forall X)(\psi(X))$ is true over no $R(\alpha+2)$; choose $\{A_\alpha\}_{\alpha<K} \in \prod_{\alpha<K} R(\alpha+2)$ such that $R(\alpha+1) \models \neg\psi(A_\alpha)$ for each α. Then as

$$\prod_{\alpha<K} \langle R(\alpha+1), \varepsilon, A_\alpha\rangle \cong \langle R(K+1), \varepsilon, A\rangle$$

where $A = e_2(\{A_\alpha\}_{\alpha<K})$, by the fundamental theorem of ultraproducts $R(K+1) \models \neg\psi(A)$. This is a contradiction. (In fact, this proof shows that a π_1^2-sentence reflects to μ-measure 1 many α's.) $\square$

Theorem 3.2.5. *The least completely ineffable cardinal is Δ_1^2 describable. In fact, there are sentences $\varphi_1 \in \Sigma_1^2$ and $\varphi_2 \in \pi_1^2$, such that $(\forall\alpha)$ (α is completely ineffable $\Leftrightarrow R(\alpha) \models \varphi_1 \Leftrightarrow R(\alpha) \models \varphi_2$).*

Proof. φ_1 is the formalization of "An excellent class of flips exists". φ_2 is the formalization of "For no ordinal K less than $(2^K)^+$ is Q_α empty". Recall that Q_α is the αth stage in the construction of Q, the class of excellent flips. (See remarks following proof of 2.2.1.) Ordinals less than $(2^K)^+$ are coded by elements of $R(K+2)$. $\square$

Corollary 3.2.6. *There is a completely ineffable below the first measurable cardinal. Actually, if μ is a normal measure on K, $\mu(\{\alpha < K \mid \alpha$ is completely ineffable$\}) = 1$.*

3.3. Weakly compacts

The set $\prod_{\alpha<K} R(\alpha+i)$ has 2^K-many elements, so in order to collapse it to a nice structure we need to compare the sizes of 2^K-many subsets of K. A cardinality-K sized subset of $\prod_{\alpha<K} R(\alpha+i)$ would only require a measure that measured K-many subsets of K to do the job. The flips for weakly compact cardinals provide just such a piece of an ultrafilter. If $C \subseteq 2^K$ with $\bar{\bar{C}} = K$ and $\tilde{C} \sim C$ such that for each $\mathscr{C} \subseteq \tilde{C}$ with $\bar{\bar{\mathscr{C}}} < K$ we have $(\bigcap \mathscr{C})^- = K$, then we can associate with $\tilde{C}$ a measure μ on C, by $\mu(A) = 1$ if $A \in \tilde{C}$ and $\mu(A) = 0$ if $K - A \in \tilde{C}$. It is easy to see that μ trivially extends over the K-complete subalgebra of 2^K generated by C, and that μ is K-additive on this subalgebra.

We would like to mimic the indescribability proof for measurable cardinals. In the course of the argument, it will be necessary to examine the set, B, of α for which $R(\alpha) \models \varphi$, where φ is part of the sentence we are trying to reflect. B is the set of α for which an arithmetic sentence (a sentence which is Δ_n^0 for some n) is true in $R(\alpha)$, and so B is itself arithmetic (has an arithmetic definition) in $R(K)$. Thus, if we include all arithmetic subsets of K in the scope of our partial measure μ, we will insure that μ can measure sets of B's ilk. Also, in order to insure that the fundamental theorem of ultraproducts holds of our eventual structures, it will be necessary to choose a witness for an event in each $R(\alpha)$ so that we may put together a witness for that event in $R(K)$. So we will have to include enough subsets of K to provide the necessary choice functions.

Definition. Let K be weakly compact and $X \subsetneq R(K)$. Let $\mathscr{I}$ be some choice function on $R(K)(\mathscr{I}(x) \in x$ for all $x \in R(K)$ such that $x \neq \emptyset)$ with $\mathscr{I}(\emptyset) = \emptyset$, and let $G \subsetneq R(K)$ be the graph of $\mathscr{I}$; $G = \{\langle x, y\rangle \mid \mathscr{I}(x) = y\}$. We define ${}_X\mathscr{A}$ to be the collection of those subsets of $R(K)$ which are bold-faced arithmetic in $X \times G$. That is ${}_X\mathscr{A}$'s elements have defining formulas in π^0_n with $X \times G$ as the parameter. Define[13] ${}_X\mathscr{P}_{\alpha<K} R(\alpha + i)$ as those elements of $\prod_{\alpha<K} R(\alpha + i)$ whose graphs are in ${}_X\mathscr{A}$. Let ${}_X\tilde{\mathscr{A}}$ be a flip with the desired property and ${}_X\mu$ be the associated measure on the K-sized, K-additive subfield of 2^K generated by ${}_X\mathscr{A}$; the X-subscript will henceforth be dropped in referring to the partial measure. ${}_X\mathscr{P}_{\alpha<K}\langle R(\alpha + i), \varepsilon\rangle/\mu$ is defined in the obvious way, analogously to the ultraproduct for measurables.

Lemma 3.3.1. *With K, μ as above,*

$$ {}_X\mathscr{P}_{\alpha<K}\langle R(k(\alpha)), \varepsilon\rangle/\mu \cong \langle R(K), \varepsilon\rangle, $$

for some $k \in {}_X\mathscr{P}_{\alpha<K} R(\alpha)$, $k: K \to K$ with $k(\alpha) \leq \alpha$ for all $\alpha < K$.

Comment: k serves the role of the identity function, for we cannot hope that μ is in any way normal.

Proof. Let

$$ \mathscr{O} = \{[f] \in {}_X\mathscr{P}_{\alpha<K} R(\alpha + 1)/\mu \mid f(\alpha) \text{ is an ordinal for every } \alpha < K\}. $$

Define the total linear ordering $\prec$ on $\mathscr{O}$ by $[f] < [g]$ iff $\mu(\{\alpha \mid f(\alpha) < g(\alpha)\}) = 1$.

Claim. $\prec$ is a well ordering. (The proof of the claim is as in well-foundedness proofs for ultraproducts via true measures; an ω-descending chain in $\mathscr{O}$ would, by the $\aleph_1$-additivity of μ, yield an ω-descending chain of ordinals.) Now if we set $C \subsetneq \mathscr{O}$ to be those elements which are almost-everywhere constant (C is a $\prec$-initial segment), we can let $[k]$ = the least upper bound of C.

The role of identity function is played by k, for it is immediate, as in Lemma 3.2.1 for measurables, that if $[f] \in {}_X\mathscr{P}_{\alpha<K} R(k(\alpha))/\mu$ then f is almost everywhere constant. The proof continues as in Lemma 3.2.1. $\square$

Lemma 3.3.2. *With K, μ as before, there is a* 1:1 *map*

$$ e_1: {}_X\mathscr{P}_{\alpha<K}\langle R(k(\alpha) + 1), \varepsilon\rangle/\mu \to \langle R(K + 1), \varepsilon\rangle. $$

[13] The original "ultraproducts" were built from partial products. See (Skolem [34]).

e_1 is an extension of the isomorphism of the previous Lemma, and is an embedding of structures, i.e., if $e_1([f]) = A \in R(K+1)$, then ${}_X\mathscr{P}_{\alpha<K}\langle R(k(\alpha)), f(\alpha), \varepsilon\rangle/\mu \cong \langle R(K), A, \varepsilon\rangle$.

Proof. e_1 is defined as in Lemma 3.2.2. □

Note, this fact is parallel to Lemma 3.2.3 for measurables, but at one level down. e_1 is not onto (some good flips are not in its range).

Lemma 3.3.3. *${}_X\mathscr{P}_{\alpha<K}\langle R((k(\alpha)), \varepsilon, f(\alpha)\rangle/\mu$ satisfies the fundamental theorem of ultraproducts, where $f \in {}_X\mathscr{P}_{\alpha<K} R(k(\alpha)+1)$.*

Proof. The fundamental theorem of ultraproducts is usually proven by induction on complexity of formulas. The only non-trivial step is in showing if $(\exists Y)(\psi(Y))$ is true in almost every $\langle R(k(\alpha)), \varepsilon, f(\alpha)\rangle$, it is true in ${}_X\mathscr{P}_{\alpha<K}\langle R(k(\alpha)), \varepsilon, [f]\rangle/\mu$. For each α let

$$w_\alpha = \{Y \in R(k(\alpha)) \mid \langle R(k(\alpha), \varepsilon, f(\alpha)\rangle = \psi(Y)\}.$$

Then the sequence $\{w_\alpha\}_{\alpha<K}$ is in ${}_X\mathscr{P}_{\alpha<K} R(k(\alpha)+1)$ so $\{\mathscr{J}(w_\alpha)\} \in {}_X\mathscr{P}_{\alpha<K} R(k(\alpha))$. By the inductive hypothesis, since $R(k(\alpha)) \models \psi(\mathscr{J}(w_\alpha))$ for almost every

$$\alpha, {}_X\mathscr{P}_{\alpha<K}\langle R(k(\alpha)), \varepsilon, [f]\rangle/\mu \models \psi([\{\mathscr{J}(w\alpha)\}_{\alpha<K}]),$$

hence ${}_X\mathscr{P}_{\alpha<K}\langle R(k(\alpha)), \varepsilon, f(\alpha)\rangle/\mu \models (\exists x)(\psi(x))$. □

Theorem 3.3.4. *A weakly compact cardinal is π^1_1-indescribable.*

Proof. Let $R(K) \models \varphi$, where φ is the sentence $(\forall Y(\psi(Y, Z))$. Here $Z \subseteq R(K)$ is φ's parameter and ψ is arithmetic. Say $(\forall\alpha) < K$, $R(\alpha) \not\models \varphi$ and let $A_\alpha \subseteq R(\alpha)$ for each α, with

$$\langle R(\alpha), \varepsilon, Z \cap R(\alpha)\rangle \models \neg\psi(A_\alpha, Z \cap R(\alpha)).$$

Set $X = \{\langle\alpha, x\rangle \mid x \in A_\alpha\} x Z$. X codes both Z and $\{A_\alpha\}_{\alpha<K}$. Form

$${}_X\mathscr{P}_{\alpha<K}\langle R(k(\alpha)+1), \varepsilon\rangle/\mu.$$

Now k, our pseudo-identity, is arithmetic, so the function $f_1(\alpha) = A_{k(\alpha)}$ is in our partial product, as is $f_2(\alpha) = Z \cap R(k(\alpha))$. Set $f(\alpha) = f_1(\alpha) \times f_2(\alpha)$.

$$\mu(\{\alpha \mid \langle R(k(\alpha)), \varepsilon, f(\alpha)\rangle \models \psi(f_1(\alpha), f_2(\alpha))\}) = 1,$$

so by the fundamental theorem

$${}_X\mathscr{P}_{\alpha<K}\langle R(k(\alpha)), \varepsilon, f(\alpha)\rangle \models \psi([f_1], [f_2])$$

so $R(K) \models \psi(e_1([f_1]), Z$. $R(K+1) \not\models (\forall Y)(\psi(Y, Z))$. Contradiction. □

Corollary 3.3.5. *There are strong inaccessibles beneath any weakly compact.*

Proof. It is easy to see that "K is strongly inaccessible" is π^1_1 over $R(K)$, hence the statement reflects. $\square$

3.4. Ineffables

The argument showing that ineffables are π^1_2-indescribable is similar to the indescribability argument for weakly compacts, but the flip used to construct the partial ultraproduct, $\mathcal{P}_c\langle R(\alpha+1), \varepsilon\rangle/\mu$ has some "normality" properties that allow us to handle one more quantifier.

Definition. If K is an ineffable cardinal and $\tilde{C}$ is a flip on $C \subseteq 2^K$, $\bar{\bar{C}} = K$, such that the diagonal intersection, $\Delta\tilde{C}$, is stationary, we can define a *measure* μ on a certain subfield of 2^K by $\mu(A) = 1$ iff $A \subseteq \Delta\tilde{C} \cap F$ for F some closed, unbounded subset of K.

(The subfield is exactly sets of the form $\Delta\tilde{C} \cap F$ and their complements.) Now μ is closed under diagonal intersection in that the diagonal intersection of any K-many μ-large sets is μ-large, and of course any element of the original collection has measure 1 iff it was not flipped. We will call such a μ "good".

A sketch of a possible indescribability argument is as follows. When we glue together witnesses A_α, to the failure of a π^1_2-truth in each $R(\alpha)$, we get a set A that, hopefully, witnesses its failure in $R(K)$. The statement, ψ, that A *does* witness its failure is π^1_1 over $R(K)$ with A as a new parameter. Now we take $\Delta\tilde{C}$ by viewing $\tilde{C}$ as a sequence, taken in any order, (the order won't matter), expand the structure to include things arithmetic in A, using only K's weak compactness this time. Finally, apply Σ^1_1-reflecting arguments for weakly compacts to show that $\psi(A)$ is true since $\psi(A \cap R(\alpha))$ is true almost everywhere. But we really only know that $\psi(A_\alpha)$ is true. Fortunately, μ's normality properties show us that, most of the time $A_\alpha = A \cap R(\alpha)$.

A cleaner proof, the one we shall use, replaces the last step of the above with the observation, that if K is strongly inaccessible, and θ is Σ^1_1 over $R(K)$ with a parameter X and $R(K) \models \theta$ then

$$S_\theta = \{\alpha < K \mid \langle R(\alpha), X \cap \alpha\rangle \models \theta\}$$

is a closed unbounded subset of K. If we let θ be ψ, and $X = A$, we get that $\mu(S_\theta) = 1$ so there are measure 1-many α with $R(\alpha) \models \psi(A)$, a contradiction.

The advantage of the first proof is that it suggests a way of proving that a cardinal which always has n-good flips is π^1_{n+1}-indescribable. A 1-good flip is a flip having the diagonal intersection property. An $n+1$-good flip is a flip which, given any additional $C \subseteq 2^K$, $\bar{\bar{C}} = K$, has an extension covering C which is an n-good flip. The proof would work by expanding the partial ultraproduct one time for each addition quantifier, and using measures which extend each other as they become poorer. Note, in reference to the section on complete ineffability to come, that it is enough to have n-good flips for each $n \in \omega$, to get π_n^1-indescribability for all n, but this property is considerably weaker than complete ineffability. With this outline in mind, here is the argument.

Lemma 3.4.1. *Let K be ineffable. Then for a good μ*

$${}_X\mathcal{P}_{\alpha<K}\langle R(\alpha), \varepsilon\rangle/\mu \cong R(K).$$

Proof. As in Lemma 3.3.1 for weakly compacts, but we must show that the pseudo-identity function, k, is now the identity; that is, if $f: K \to K$ with $f(\alpha) < \alpha$ for all $\alpha < K$, $f \in {}_X\mathcal{P}_{\alpha<K}\langle R(\alpha), \varepsilon\rangle$, then f is constant almost everywhere. But if not, then for each α the set $B_\alpha = \{\beta \mid f(\beta) \neq \alpha\}$ is measure 1 by μ. If $B = \Delta_{\alpha<K} B_\alpha$ is the diagonal intersection of the B_α's, then for any $\gamma \in B$, $f(\gamma) \not< \gamma$, contradicting $f(\alpha) < \alpha$ for all $\alpha < K$. (All we really need is for B to be non-empty; we have $\mu(B) = 1$.) $\square$

Lemma 3.4.2. *There is a one-to-one map*

$$e_1: {}_X\mathcal{P}_{\alpha<K} R(\alpha+1)/\mu \to R(K+1),$$

that is an embedding of structures, i.e., if $e_1([f]) = A \subseteq K$, and $f(\alpha) = A_\alpha \subseteq R(\alpha)$, then

$${}_X\mathcal{P}_{\alpha<K}\langle R(\alpha), A_\alpha, \varepsilon\rangle/\mu \cong \langle R(K), A, \varepsilon\rangle.$$

Also, $\mu(\{\alpha \mid A \cap R(\alpha) = A_\alpha\}) = 1$.

Proof. First part as earlier in Lemma 3.3.2. The "Also" is the important part. For convenience we suppose $A \subseteq K$. For each $\alpha \in K$, $B_\alpha = \{\gamma \mid \alpha \in A_\gamma \iff \alpha \in A\}$ has measure 1, so $B = \Delta_{\alpha<K} B_\alpha$ has measure 1. But for any $\delta \in B$, $A \cap R(\delta) = A_\delta$. $\square$

Lemma 3.4.3. *For K strongly inaccessible, $X \subseteq K$, and $\psi(X)$ a Σ_1^1 sentence true over $R(K)$,*

$$\{\alpha \mid \langle R(\alpha), \varepsilon, X \cap R(\alpha)\rangle \models \psi(X)\}$$

is closed and unbounded.

Proof. It is a well known result that $\{\alpha \mid R(\alpha)$ is an elementary substructure of $R(K)\}$ is closed and unbounded; to extend to Σ_1^1 formulas, observe footnote 11. $\square$

Theorem 3.4.4. *If K is ineffable, K is π_2^1-indescribable.*

Proof.[14] Say $\varphi = (\forall X)(\exists Y)(\psi(X, Y))$ is true in $R(K)$, but not in any $R(\alpha)$ for $\alpha < K$. For each α choose $A_\alpha \subseteq R(\alpha)$, such that $R(\alpha) \models (\forall Y) \neg \psi(A_\alpha, Y)$ and let

[14] This time we've completely dropped mention of φ's parameter.

X code the $\{A_\alpha\}$ sequence. Form ${}_X\mathcal{P}_{\alpha<K}\langle R(\alpha+1), \varepsilon\rangle/\mu$ and let $A = e_1([f])$, where $f(\alpha) = A_\alpha$. Then by the Fundamental Theorem

$${}_X\mathcal{P}_{\alpha<K}\langle R(\alpha+1), f(\alpha), \varepsilon\rangle/\mu \models (\forall Y)\neg\psi[f], Y);$$

we would like

$$\langle R(K+1), A, \varepsilon\rangle \models (\forall Y)\neg\psi(A, Y)$$

in order to gain a contradiction. But if $\langle R(K), A, y_0, \varepsilon\rangle \models (\exists Y)(\psi(A, Y))$, by Lemma 3.4.3 there is a closed and unbounded set F for which $\alpha \in F \Longrightarrow R(\alpha) \models (\exists Y)(\psi(A \cap \alpha, Y))$. Of course, $\mu(F) = 1$ and

$$\mu(\{\alpha \mid A_\alpha = A \cap R(\alpha)\}) = 1$$

so

$$\mu(\{\alpha \mid R(\alpha) \models (\exists Y)\psi(A_\alpha, Y)\}) = 1,$$

a contradiction. $\square$

Corollary 3.4.5. *There are weakly compacts and weakly ineffables beneath any ineffable cardinal.*

Proof. There are π_1^1 definitions of weak compactness and weak ineffability. $\square$

3.5. *Completely ineffables*

We could, using techniques similar to those used previously, show that for K completely ineffable, K is π_n^1-indescribable for all n. A "tight" indescribability argument, however, yields a class Σ (of sentences which reflect) such that the set, $\Sigma^{\neg}$, of negations of elements of Σ contains a sentence φ which describes the cardinal. Since $(\bigcup_{n\in\omega}\pi_n^1)^{\neg} = \bigcup_{n\in\omega}\pi_n^1$ it is clear that $\bigcup_{n\in\omega}\pi_n^1$ is not the natural, maximal indescribability class for completely ineffables. We shall show that if $\mathcal{A}_{K+1}$ is the least (standard, transitive) admissible set containing $R(K+1)$ as an element, and if φ is Σ_1 over $\mathcal{A}_{K+1}$, $\mathcal{A}_{K+1} \models \varphi$, then $\mathcal{A}_{\alpha+1} \models \varphi$ for some $\alpha \in K$. (By Σ_1 we mean the class of φ in prenex form with at most one unlimited existential quantifier interpreted as ranging over elements of $\mathcal{A}_{K+1}$; other quantifiers are limited, and φ may have constants from $\mathcal{A}_{K+1}$, but has no special predicates denoting subsets of $\mathcal{A}_{K+1}$.) There is a $\psi \in \pi_1$ such that $\mathcal{A}_{K+1} \models \psi \iff K$ is completely ineffable.

Previously we worked with flips; i.e., pieces of a measure that lie within a model. For this argument we will need the complete outside normal measure on K, so we will have to mention a countable, standard, transitive model, M, satisfying ZFC + "K is completely ineffable." For the remainder of Section 3.5, K and M will be such objects. By $\mathcal{A}_{\alpha+1}$ we will mean M's $\mathcal{A}_{\alpha+1}$. When we state theorems about K we mean that the statements hold about κ in M. However, by the usual techniques, these statements can be translated into theorems in ZFC + "K is completely ineffable". (Here is a brief description of the translation procedure: say we prove a statement σ about K in M. Argue instead as follows: Suppose σ fails. Now take M to be a sufficiently elementary countable submodel of the universe. In particular, $M \models \neg\sigma$. Force over M to get an outside normal measure. Now repeat the argument proving σ in M, a contradiction. Thus σ must actually hold.)

Lemma 3.5.1. *Let $M^K = \{f: K \to M \mid f \in M\}$ and let $\mathcal{U}$ be a normal M-ultrafilter on K. Let $\mu(X) = 1$ if $X \in \mathcal{U}$, $\mu(X) = 0$ if $K - X \in \mathcal{U}$. Then S, defined to be the standard part of M^K/μ, contains an isomorphic copy of the least admissible set containing $R(K+1)$ as an element.*

Proof. Claim. M^K/μ is admissible.

Proof of claim. By the fundamental theorem of ultraproducts, easily verified in these circumstances, $M^K/\mu \equiv M$, so $M^K/\mu \models \mathrm{ZFC}$.

Claim. Let S be the well founded part (standard part) of M^K/μ. Then there is a naturally defined embedding e, of $R(K+1)$ into S.

Proof of claim. The proof is similar to that for previous ultraproduct arguments. For $\alpha < K$, the constant function $[f_\alpha]$ becomes identified with α, and the identity function $\mathrm{id}(\alpha) = \alpha$ is sent via e to K (use normality) so that $\Pi_{\alpha<K} R(\alpha)/\mu$, considered as a subset of M^K/μ, is identified with $R(K)$ and $\Pi_{\alpha<K} R(\alpha+1)/\mu \cong R(K+1)$, as in the measurable cardinal case.

Now, as $K \in S$ and $M^K/\mu \models$ the power set axiom, $(2^K)^M \in S$. The standard part of an admissible structure is admissible[15], so $\mathcal{A}_{K+1}$, the least admissible set containing $R(K+1)$ as an element is contained in S. $\square$

Lemma 3.5.2. *For $[f] \in M^K/\mu$, if $[f] \in \mathcal{A}_{K+1}$ (considered as embedded in M^K/μ), then $f(\alpha) \in \mathcal{A}_{\alpha+1}$ almost everywhere.*

Proof. If $[f] \in \mathcal{A}_{K+1}$, then $M \models$ "$[f]$ is in the least admissible set containing $R([g]+1)$" where g is the identity function. Apply the fundamental theorem. $\square$

Theorem 3.5.3. *Let $\varphi([f_1],[f_2],\ldots,[f_n])$ be a limited formula over $\mathcal{A}_{K+1}$. Then $\varphi(f_1(\alpha), f_2(\alpha),\ldots,f_n(\alpha))$ is true over $\mathcal{A}_{\alpha+1}$ for almost every α iff $\mathcal{A}_{K+1} \models \varphi([f_1],\ldots,[f_n])$.*

Proof. By induction on complexity of φ, as in fundamental theorem of ultrapowers. The only non-trivial step (and the one that shows why φ's quantifiers must be limited), is that if $\mathcal{A}_{\alpha+1} \models (\exists x \in f(\alpha))\,\psi(x, f(\alpha))$ almost everywhere, then $\mathcal{A}_{K+1} \models (\exists x \in [f])\,\psi(x,[f])$, assuming that sentences of ψ's complexity satisfy the

[15] See (Barwise [2]).

theorem. Simply choose $g(\alpha) \in f(\alpha)$, so that almost everywhere $\mathscr{A}_{\alpha+1} \models \psi(g(\alpha), f(\alpha))$. Now $M^K/\mu \models [g] \in [f]$ so, since $\mathscr{A}_{K+1}$ is transitive, $[g] \in \mathscr{A}_{K+1}$, so $\mathscr{A}_{K+1} \models \psi([g],[f])$, so

$$\mathscr{A}_{K+1} \models (\exists x \in [f])\, \psi(x,[f]). \quad \square$$

Had the existential quantifier been unlimited, the witness we glued together might not have been in $\mathscr{A}_{K+1}$.

Corollary 3.5.4. *Let $\psi = (\exists x)\, \varphi(x,[f_1],\ldots,[f]_n)$ be such that $\mathscr{A}_{K+1} \models \psi$. Then $\mathscr{A}_{\alpha+1} \models (\exists x)\, \varphi(x, f_1(\alpha),\ldots,f_n(\alpha))$ for almost every α. So K is Σ_1-over-the-next-admissible indescribable.*

Proof. The old "get an existential quantifier for free" trick. If $\mathscr{A}_{K+1} \models \psi$, then $\mathscr{A}_{K+1} \models \varphi([g],[f_1],\ldots,[f_n])$, so by Theorem 3.5.3, $\mathscr{A}_{K+1} \models \varphi(g(\alpha), f_1(\alpha),\ldots,f_n(\alpha))$ almost everywhere, so $\mathscr{A}_{\alpha+1} = \psi$ almost everywhere.
N.B. If $[f] \in R(K+1)$ viewed as embedded in the ultraproduct, then $f(\alpha) = [f] \cap R(\alpha)$ almost everywhere. $\square$

Corollary 3.5.5. *K is π^1_n-indescribable for all n.*

Proof. A π^1_n-sentence is a limited sentence over $\mathscr{A}_{K+1}$ since $R(K+1) \in \mathscr{A}_{K+1}$. $\square$

Theorem 3.5.6. *There is a sentence ψ which is π_1 over $\mathscr{A}_{\alpha+1}$ such that $\mathscr{A}_{\alpha+1} \models \psi \iff \alpha$ is completely ineffable.*

Proof. ψ simply says that for all α in $\mathscr{A}_{\alpha+1}$ $Q_\alpha \neq \emptyset$. (See Corollary 2.2.4.) $\square$

We have now exactly characterized the indescribability inherent in complete ineffability.

Our immediate inference is

Corollary 3.5.7. *There are ineffable cardinals less than K.*

Proof. "K is ineffable" has a π^1_3-definition, so it reflects

(K is ineffable $\iff (\forall t \in S_K)(\exists t' \sim t)(\forall$ closed unbounded $C)\{C \cap \Delta t' \neq \emptyset\}$). $\square$

An interesting fact about the construction of Q can now be seen.
Let us define γ to be the least ordinal not in $\mathscr{A}_{K+1}$. In Fact 2.2.2, we showed that $\theta \leq \gamma$, where θ = number of steps required to construct Q. The reflection property for complete ineffables shows that $\theta \not< \gamma$, for if $\theta < \gamma$, then $\mathscr{A}_{K+1} \models (\exists\theta)[Q_{\theta+1} = Q_\theta \neq \emptyset]$. So some $K_1 < K$ has $\mathscr{A}_{K_1+1} \models (\exists\theta)[Q_{\theta+1} = Q_\theta \neq \emptyset]$.

But this means that K_1 is completely ineffable, so the statement reflects to a $K_2 < K_1$, and so on, yielding a descending chain of ordinals.

So, to check if a cardinal K is completely ineffable, we need only carry out the process of squeezing down on Q for $\gamma + 1$-many steps. Were K not completely ineffable this fact would be discovered in less than-γ many steps. If κ is completely ineffable, we will know this at exactly the γth step. This answers a question of Baumgartner.

Remarks on the forcing arguments of Section 2

It is possible that (if K is actually measurable) the construction of Q as above will yield a measure? I.e., could it happen that $\bigcup Q$ be a measure? No, since this means that for every $A \subseteq K$ either A or $K - A$ would disappear from some $\bigcup Q_\alpha$, so by admissibility there would be an $\alpha < \gamma$ such that $\bigcup Q_\alpha$ is a measure, which by the reflection property for completely ineffables leads to an infinite descending chain. By repeating this argument "above a fixed $\mathscr{C} \in Q$", we see that $\bigcup \{\mathscr{D} \in Q \mid \mathscr{D} \geqslant \mathscr{C}\}$ cannot be a measure either. Thus the forcing of Section 2, which yields a normal M-ultrafilter, always yields a generic object outside the ground model (as long as we use the canonically constructed Q as forcing conditions). This fact is, of course, not surprising and could be proved just using the indescribability property of measurable cardinals. But the type of analysis used above answers a more interesting question. If a weakly compact cardinal actually is completely ineffable, could the forcing we used to produce an M-ultrafilter happen to produce a normal M-ultrafilter?

This too is impossible. Consider the forcing argument in Theorem 2.1.2. Suppose $\mathscr{U}$ were a normal M-ultrafilter. Then $M[G] \models$ "$\bigcup G$ is a normal M-ultrafilter". But the statement in quotes can be written as a first-order formula, $\varphi(G)$, where all quantifiers are restricted to $(2^K)^M$. i.e., as a formula of rank $K + 1$ in the language of forcing. Now some $\mathscr{C} \Vdash \varphi(G)$. By using the syntactic definition of "$\Vdash$" together with the fact that $(2^K)^M \in \mathfrak{A}_{K+1}$, we can find a formula Ψ, Σ_1 over $\mathfrak{A}_{K+1}$, saying that some condition $\Vdash \varphi(G)$. But now by reflecting Ψ to an ordinal below K we get another ordinal, which must be completely ineffable (since it satisfies Ψ), and continuing to reflect Ψ we once again get an infinite descending chain.

3.6. A little coherence goes a long way[16]

There is one comparison for which the previous approach is insufficient. Both weakly compact and weakly ineffable cardinals are π^1_1 indescribable and π^1_2 describable. But by using a small part of the coherence strength of weakly ineffables we shall soon see that below any such cardinal there is a K which is π^n_m-

[16] The results in this section were already known and although our proofs of them do not involve flipping properties, we include them for the sake of completeness.

indescribable for all n and m. It is well-known that any π_1^1-indescribable is weakly compact, so there are weakly compacts beneath the first weakly ineffable.

The outline of the argument is as follows: we shall code a "description" of α as a subset, A, of α. That is, A_α has some property true about $R(\alpha)$ such that $A_\alpha \cap \gamma$ fails to have the property for $R(\gamma)$, wherever $\gamma < \alpha$. Therefore, if $\alpha < \beta$, A_α and A_β cannot possible cohere. There is a hitch, in that the codings only work well for certain α's. The set of "nice" α's turns out to be closed and unbounded, so the following lemma surmounts the coding problem.

Lemma 3.6.1. *Let K be weakly ineffable and F be a closed, unbounded subset of K. Say $A_\delta \subseteq \delta$ for each $\delta < K$. Then there are two ordinals, α and β, in F with $\alpha < \beta$ and $A_\alpha = A_\beta \cap \alpha$.*

Remark. In fact, the proof shows there is a K-sized set $B \subseteq F$, with $A_\eta = A_\gamma \cap \eta$ for any $\eta, \gamma \in B$ with $\eta < \gamma$, but the lemma as stated is all we need. A cardinal which has the property of the lemma (coherence on any two points of a closed, unbounded set) is called a subtle cardinal.[17]

Remark. This lemma also shows why the step from weakly ineffable to ineffable is one of introducing an additional uniformity; the coherence definition for ineffable states there is a single set of coherence, which intersects all closed, unbounded sets; a weakly ineffable might need different sets of coherence to meet different closed, unbounded sets.

Proof of lemma. Define $A^*_\alpha \subseteq \alpha$ for each $\alpha < K$ by $A^*_\alpha = A_\eta$, where $\eta = \sup(F \cap (\alpha \cup \{\alpha\})$. This doesn't define A^*_α's for those $\alpha < \inf(F)$; we can just let such A^*_α's be null. Note for $\alpha \in F$, $A^*_\alpha = A_\alpha$.

By weak ineffability there is a $B \subseteq K$, $\bar{\bar{B}} = K$ on which the A^*_α's cohere. Choose $\delta, \eta \in B$ so that there is at least one $\gamma \in F$ with $\inf F < \delta < \gamma < \eta$. Then there is an $\alpha \leq \delta$ and a $\beta \leq \eta$, distinct elements of F, such that $A^*_\delta = A_\alpha$ and $A^*_\eta = A_\beta$. Since $A^*_\delta = A^*_\eta \cap \delta$, $A_\alpha = A_\beta \cap \alpha$. □

Corollary 3.6.2. *Let K, F be as above. Say $A_\delta \subseteq R(\delta)$ for each $\delta \in F$. Then there are two ordinals, α and β, in F with $\alpha < \beta$ and $A_\alpha = A_\beta \cap R(\alpha)$.*

Proof. (A simple coding). Let $F' = F \cap \{\alpha < K \mid (R(\alpha))^= = \alpha\}$. F' is still closed unbounded. Let c map $R(K)$ 1–1 onto K in such a way that whenever $(R(\alpha))^= = \alpha$, $c''R(\alpha) = \alpha$. Let

$$A'_\delta = \begin{cases} \emptyset & \text{if } \delta \notin F' \\ c''A_\delta & \text{if } \delta \in F'. \end{cases}$$

[17] For more on subtle cardinals see Jensen–Kunen [13].

For $\alpha, \beta \in F'$, if $A'_\alpha = A'_\beta \cap \alpha$, then $A_\alpha = A_\beta \cap R(\alpha)$. $\square$

Theorem 3.6.3. *Let K be weakly ineffable. Then there is a cardinal $\alpha < K$ such that α is π^m_n indescribable for all m and n.*

Proof. Say not. Let $F = \{\alpha < K \mid (R(\alpha))^* = \alpha\}$. For each $\alpha \in F$ let $\varphi_\alpha(X)$ be a $\pi^{m_\alpha}_{n_\alpha}$ formula, and $X_\alpha \subseteq R(\alpha)$ be such, that $\langle R(\alpha), X_\alpha, \varepsilon\rangle \models \varphi(X)$ but for each $\gamma < \alpha$,

$$\langle R(\gamma), X_\alpha \cap R(\gamma), \varepsilon\rangle \models \neg\varphi(X).$$

Let $p_\alpha \in \omega$ be the Gödel number (in some fixed numbering scheme for formulas) of $\varphi_\alpha(X)$ and let

$$A_\alpha = \{(0, p_\alpha)\} \cup \{(1, m_\alpha)\} \cup \{(2, n_\alpha)\} \cup \{(3, x) \mid x \in X_\alpha\}.$$

But given $\alpha, \beta \in F$, $\alpha < \beta$, if $A_\alpha = A_\beta \cap R(\alpha)$, we get $\varphi_\alpha(X)$ to be the same formula as $\varphi_\beta(X)$, $m_\alpha = m_\beta$, $n_\alpha = n_\beta$, and $X_\alpha = X_\beta \cap R(\alpha)$, yielding an immediate contradiction. $\square$

4. Absoluteness to L

4.1.

The flipping characterizations of the cardinals we have studied make it easy to show that they are absolute to L. The key observation is that if a K-sequence in L has a flip in V with a large diagonal intersection, all initial segments of the flip are in L. In fact, what the flip does to the first α-many sets can be *read off from any single ordinal in the intersection of the first α-many sets in the flip.*

Theorem 4.1.1. *Let $P(K)$ be any of the following properties*
(i) *K is strongly inaccessible*
(ii) *K is weakly compact*
(iii) *K is weakly ineffable*
(iv) *K is ineffable*
(v) *K is completely ineffable.*
Let P^L be the relativization of P to the class of constructible sets. Then $P(K) \Longrightarrow P^L(K)$.

Proof. The same technique proves cases (i), (ii), (iii) and (iv). For concreteness, and since this case has the largest number of details to check, we take $p(K)$ to be K is ineffable. We have that $(\forall t \in S_K)(\exists t')[t' \sim t$ and $\Delta t'$ is stationary]. We need to show that in L, K has this same property. So let $t \in L$, $t: K \to 2^K$. Let t' be such that $t' \sim t$ and $\Delta t'$ is stationary. We will show that t' itself is in L. Then we are done since $(\Delta t')^L = \Delta t'$ and since any subset of K which is closed in L is truly closed, $\Delta t'$ is stationary in L.

Showing $t' \in L$ is, plainly, equivalent to showing that for

$$T' = \{\alpha < K \mid t'(\alpha) = t(\alpha)\}, \quad T' \in L.$$

Notice that for any $\alpha < K$, $T' \upharpoonright \alpha \in L$ since, and here is the main observation, if we take any $\delta > \alpha$ with $\delta \in \Delta t'$, then $T' \cap \alpha = \{\beta < \alpha \mid \delta \in t(\beta)\}$. (The corresponding proof for the case $P(K)$ = "K is strongly inaccessible" ends right here.) So we have $T' \subseteq K$ with the property that $\forall \alpha < K[T' \cap \alpha \in L]$. Since K is π_1^1 reflecting, the following lemma will get the result.

Lemma 4.1.2. *There is a π_1^1 predicate $\psi(A)$ such that for any ordinal $\delta > 0$, $R(\delta) \models \psi(A)$ iff δ is a regular cardinal and A is a non-constructible subset of δ.*

Proof of Lemma. $\psi(A)$ says that

(i) Every class which codes a function $f: On \to On$ satisfies $(\forall \alpha)(\exists \beta)[\beta \supseteq f''\alpha]$.

(ii) Every class which codes a wellfounded model, $\langle M, E \rangle$, of $V = L$ does not have A as an element.

Since (i) guarantees that whenever $\delta > 0$ and $R(\delta) \models \psi(A)$, δ will be a regular cardinal, to say "wellfounded" in (ii) we need only say that no x codes an infinite descending ε-chain.

Now suppose $\delta > 0$ and $R(\delta) \models \psi(A)$. Then δ is regular. If A were constructible, then for some β, $\delta < \beta < \delta^+$, $A \in L_\beta$ (= the class of sets constructed by level β). As $\bar{\bar{L}}_\beta = K$, we can code $\langle L_\beta, \varepsilon \rangle$ by some $\langle M, E \rangle$, a subset of $R(\delta)$, so $R(\delta) \not\models \psi(A)$. So A is non-constructible. Conversely, if δ is regular and A is a non-constructible subset of δ, clearly $R(\delta) \models \psi(A)$. $\square$

Now, were $T' \notin L$ we would have $R(K) \models \psi(T')$ which gives us a $\delta < K$ such that $T' \cap \delta$ is non-constructible. Thus $T' \in L$; so $t' \in L$, and we are done.

Now for the case in which $P(K)$ is "K is completely ineffable", we use a forcing argument which can, by the usual tricks, be converted to a proof in ZFC. Suppose M is a countable standard transitive model of ZFC + K is completely ineffable. As in Theorem 2.2.1, we use forcing to construct $\mathcal{U}$, a normal M-ultrafilter on K. Now all the properties of being a normal $(L)^M$-ultrafilter, except for property (iv) are routinely seen to hold for $\mathcal{U} \cap (L)^M$. The indescribability argument used for the earlier cases of this theorem shows that $\mathcal{U} \cap (L)^M$ satisfies property (iv). So by Theorem 2.2.1, $L \models K$ is completely ineffable. $\square$

References

[1] F. Abramson, Flipping Properties and Equiconsistency Results for Large Cardinals, Ph.D. Thesis, M.I.T. Dept. of Mathematics, June 1974.

[2] J. Barwise, Infinitary logic and admissible sets, J. Symbolic Logic, 34 (1969) 226–252.

[3] J. Baumgartner, Ineffability properties of cardinals, I, in: Proc. Int. Coll. on Infinite and Finite Sets, Keszthely, Hungary, June 1973.

[4] P. Cohen, Set Theory and the Continuum Hypothesis, (Benjamin, New York, 1966).
[5] P. Erdös and A. Hajnal, On the structure of set mappings, Acta Math. Acad. Sci. (Hungar.) 9 (1948) 11–131.
[6] P. Erdös, A. Hajnal and R. Rado, Partition relations for cardinal numbers, Acta Math. Acad. Sci. (Hungar.) 16 (1965) 93–196.
[7] P. Erdös and R. Rado, A combinatorial theorem, J. London Math. Soc. 25 (1950) 249–255.
[8] P. Erdös and R. Rado, Combinatorial theorems on classification of subsets of a given set, Proc. London Math. Soc. (3) 2 (1952) 417–439.
[9] P. Erdös and R. Rado, A partition calculus in set theory, Bull. Am. Math. Soc. 62 (1956) 427–489.
[10] G. Fodor, On stationary sets and regressive functions, Acta. Sci. Math. (Szeged) 27 (1966) 105–110.
[11] H. Gaifman, Pushing up the measurable cardinal, notes for 1967 AMS–ASL Summer Institute IV R.
[11.5] W.P. Hanf and D. Scott, Classifying inaccessible cardinals (abstract), Notices Am. Math. Soc. 8 (1961) page 445.
[12] T. Jech, Some combinatorial problems concerning uncountable cardinals, Ann. Math. Logic 5 (1973) 165–198.
[13] R. Jensen and K. Kunen, Some combinatorial properties of L and V, mimeographed.
[14] H.J. Keisler and A. Tarski, From accessible to inaccessible cardinals, Fund. Math. 53 (1964) 225–308.
[15] E. Kleinberg, Strong partition properties for infinite cardinals, J. Symbolic Logic 35 (1970) 410–428.
[16] E. Kleinberg, Infinitary Combinatorics, Cambridge Summer School in Mathematical Logic held in Cambridge, England, 1971 (Springer-Verlag, Berlin, 1973) 361–418.
[17] E. Kleinberg, Theory of Large Cardinals and Partition Relations, in preparation.
[18] E. Kleinberg, The equiconsistency of two large cardinal axioms, to appear in Fund. Math.
[19] E. Kleinberg, A combinatorial characterization of normal M-ultrafilters, to appear in Advances in Math.
[20] S. Kochen, Ultraproducts in the theory of models, Ann. of Math. 74 (1961) 221–261.
[21] K. Kunen, Inaccessibility properties of cardinals, Doctoral dissertation, Stanford, 1968.
[22] K. Kunen, Some applicatiaons of iterated ultrapowers in set theory, Ann. Math. Logic 1 (1970) 179–227.
[23] K. Kunen, Indescribability and the continuum, Proceedings of Symposia in Pure Mathematics, Vol. XIII, Part 1, AMS, 1971, 199–203.
[24] K. Kunen and J. Paris, Boolean extension and measurable cardinals, Ann. Math. Logic, 2 (1971) 359–377.
[25] A. Levy, The sizes of indescribable cardinals, Proceedings of Symposia in Pure Mathematics, Vol. XIII, Part 1, AMS, 1971, 205–218.
[26] A. Levy and R. Solovay, Measurable cardinals and the continuum hypothesis, Israel J. Math. 5 (1967) 234–248.
[27] A.R.D. Mathias, On a generalization of Ramsey's theorem, Notices Am. Math. Soc. 15 (1968) p. 931, Abstract ≠ 68T–E19.
[28] J. Paris, Boolean extensions and large cardinals, Doctoral dissertation, Manchester University, 1969.
[29] F.P. Ramsey, On a problem of formal logic, Proc. London Math. Soc. (2) 30 (1930) 264–286.
[30] F. Rowbottom, Large cardinals and constructible sets, Doctoral dissertation, University of Wisconsin, Madison, Wisconsin, 1964.
[31] G. Sacks, Saturated Model Theory, W.A. Benjamin, 1972.
[32] D. Scott, Measurable cardinals and constructible sets, Bull. Acad. Polon. Sci., Sér. Sci. Math. Astronom. Phys. 7 (1961) 145–149.
[33] J. Silver, Some applications of model theory in set theory, Doctoral dissertation, University of California (Berkeley), 1966.
[34] T. Skolem, Peano's axioms and models of arithmetic, Mathematical interpretations of formal systems (North-Holland, Amsterdam 1955) 1–14.
[35] R.M. Solovay, A nonconstructible Δ_3^1 set of integers, Trans. Am. Math. Soc. 127 (1967) 50–75.
[36] S. Ulam, Zur Masstheorie in der allgemeinen Mengenlehre, Fund. Math. 16 (1930) 140–150.
[37] R.L. Vaught, Indescribable cardinals, Notices Am. Math. Soc. 10 (1963) 126.